

The spacetime Penrose inequality and enclosing area

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Abstract

We prove the sharp spacetime Penrose inequality for smooth one-ended initial-data exteriors in every spatial dimension $n \geq 3$. We assume the dominant energy condition, integrable constraint densities, finite ADM limits, weak future outer trapping, the stated differentiated decay and positive enclosing area. The ADM energy-momentum is then future timelike, and its Lorentz-invariant mass satisfies the sharp bound in terms of the least enclosing area measured in the original metric. In dimensions three and four, two metric derivatives and one tensor derivative suffice, and positive enclosing area also follows from the hypotheses. The proof combines end replacement with a coupled graph and conformal deformation that makes the area loss and energy increase arbitrarily small, followed by a Riemannian Penrose comparison.

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1 The inequality and its geometric quantity

The Penrose inequality asks how much mass is required to surround a trapped region. Its origin is a test of cosmic censorship: a collapsing configuration should settle to a black hole, radiation should carry energy away, and the horizon area should not decrease. Penrose's null-shell argument made this comparison concrete [40]. An initial-data theorem must express it using only a Riemannian metric, a second fundamental form and a specified enclosing area.

The choice of area is part of the problem. A surface outside an apparent horizon can have smaller area than the horizon itself. Ben-Dov gave spherically symmetric examples satisfying the dominant energy condition for which the bound using the apparent horizon's own area fails [5]. Carrasco and Mars found a different obstruction for generalized apparent horizons: their counterexamples already have an outer area-minimizing generalized horizon [13]. Thus a mass-area statement must specify both its trapping condition and its enclosing class. We use the infimum of the areas of all enclosing hypersurfaces separating the boundary from infinity. Every component counts, including any part coincident with the boundary. The mass is the Lorentz-invariant length of the ADM energy-momentum vector. These choices lead to the following precise formulation.

1.1 Data, area and statements

Definition 1.1 (Initial-data exterior). Let $n \geq 3$ and let Ω be a smooth connected orientable n -manifold with nonempty compact smooth boundary S . The metric g and symmetric covariant two-tensor K are smooth up to S , and g is complete with S included. Outside a compact set, Ω is a single coordinate end $\{x \in \mathbb{R}^n : |x| > R_0\}$. Thus there are no additional ends hidden in the compact-complement condition.

Write R_g for scalar curvature and ∇ for the connection of g . Write $\tau = \text{tr}_g K$. We use the second-fundamental-form sign $K(Y, Z) = \mathbf{g}(\nabla_Y \mathbf{n}, Z)$ for a future unit normal $\mathbf{n}$ when the data arise from a spacelike hypersurface, where $\mathbf{g}$ and ∇ denote the spacetime metric and its Levi-Civita connection, as in [9, Equation (3)]. Define the constraint densities by

$$2\mu = R_g + (\text{tr}_g K)^2 - |K|_g^2, \quad J_i = \nabla^j (K_{ij} - (\text{tr}_g K)g_{ij}). \quad (1.1)$$

In dimension three this normalization agrees with [29, Equation (2.3)]; μ, J are 8π times the physical densities in [9, Equations (4)–(5)]. The dominant energy condition is $\mu \geq |J|_g$. We require $\mu, |J|_g \in L^1(\Omega, dV_g)$. Along S the unit normal ν points into Ω , toward the end. Its future expansion is

$$\theta_+ = H_S + \text{tr}_S K, \quad H_S = \text{div}_S \nu, \quad \text{tr}_S K = (g^{ij} - \nu^i \nu^j) K_{ij}.$$

The boundary is weakly future outer trapped when $\theta_+ \leq 0$ on every component. This is a condition on the outward future null expansion, following [4, Section 2 and Definition 7.1]; no condition is imposed on the other null expansion.

For coordinate tensors, $T = O_j(r^{-a})$ means $|\partial^\alpha T| \leq C_\alpha r^{-a-|\alpha|}$ for $|\alpha| \leq j$, where $r = |x|$. The decay orders will be specified in each theorem. Set $k = n - 1$ and $\omega = \omega_{n-1} = |\mathbb{S}^{n-1}|$. The Laplacian is $\Delta = \text{div} \nabla$ and symmetrization includes a factor $1/2$. We require the finite ADM limits

$$E = \frac{1}{2(n-1)\omega_{n-1}} \lim_{R \rightarrow \infty} \int_{|x|=R} (\partial_j g_{ij} - \partial_i g_{jj}) n_\delta^i dA_\delta, \quad (1.2)$$

$$P_i = \frac{1}{(n-1)\omega_{n-1}} \lim_{R \rightarrow \infty} \int_{|x|=R} (K_{ij} - (\text{tr}_g K)g_{ij}) n_\delta^j dA_\delta. \quad (1.3)$$

Here n_δ and dA_δ are Euclidean. If $E > |P|_\delta$, define $m = (E^2 - |P|_\delta^2)^{1/2}$.

Definition 1.2 (Enclosing hypersurfaces). An *enclosing hypersurface* is the entire intrinsic manifold boundary $\Gamma = \partial D$ of a connected smooth codimension-zero submanifold-with-boundary $D \subset \Omega$. We require D to be closed in Ω , its manifold interior to lie in $\text{int } \Omega$, and D to contain the whole sufficiently distant end. Its full boundary must be compact, smooth, embedded and two-sided. In particular, $D = \Omega$ is admissible and has boundary S . Define

$$A_{\min,g}(S) = \inf_D \text{Area}_g(\partial D). \quad (1.4)$$

The use of intrinsic boundary ensures that coincident portions of S are counted. We also call an enclosing hypersurface a cut. A cut need not be connected, trapped or minimal. The infimum need not be attained.

Theorem 1.3 (The enclosing-area Penrose inequality). *Let (Ω^n, g, K) be an exterior in Definition 1.1, with integrable constraint densities, finite ADM limits, dominant energy condition and weakly future outer trapped boundary. Suppose*

$$g - \delta = O_6(r^{-q}), \quad K = O_5(r^{-1-q}), \quad \frac{n-2}{2} < q < n-2,$$

and assume $E > |P|_\delta$ and $A_{\min,g}(S) > 0$. Then

$$m \geq \frac{1}{2} \left(\frac{A_{\min,g}(S)}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}. \quad (1.5)$$

The coefficient and exponent are sharp in each dimension $n \geq 3$.

Thus the enclosing-area form of the spacetime Penrose conjecture holds in this class. Neither outermostness nor outer area minimization of S is assumed. The second fundamental form is arbitrary, and no spin, vacuum or interior-topology hypothesis is imposed. The time-symmetric Schwarzschild–Tangherlini exterior of mass $m > 0$ has $A_{\min,g}(S) = \omega_{n-1}(2m)^{(n-1)/(n-2)}$ and attains equality.

Corollary 1.4 (Future timelikeness under the differentiated decay assumptions). *Let (Ω^n, g, K) satisfy the hypotheses of Theorem 1.3 with the timelikeness assumption $E > |P|_\delta$ omitted. Then $E > |P|_\delta$, and Equation (1.5) holds.*

The proof follows the numerical deduction in Section 2.

Theorem 1.5 (Weaker decay in dimensions three and four). *Let $n = 3$ or $n = 4$. Let (Ω^n, g, K) be the smooth one-ended exterior specified above, with integrable constraint densities, finite ADM limits, dominant energy condition and weakly future outer trapped boundary. Assume*

$$g - \delta = O_2(r^{-q}), \quad K = O_1(r^{-1-q}), \quad q > \frac{n-2}{2}.$$

Then $A_{\min,g}(S) > 0$, $E > |P|_\delta$, and (1.5) holds.

Here positive enclosing area and timelikeness are both conclusions, and no extra decay is required of $\text{tr}_g K$. The two derivative regimes concern the original data. After preparation, both lead to ends with estimates at every fixed derivative order, allowing one common deformation theorem to prove the numerical bound.

There are also designated-end formulations for finitely many ends, with different area classes in dimensions three and four. In the three-dimensional complete-data formulation the chosen outer region may retain other ends; in four dimensions its connected outer domain excludes their distant tails. Sections 7 and 8 give their precise definitions and proofs. Their reduction to the one-ended inequality compares the two enclosing infima in the needed direction; it does not identify them. The one-ended statements above isolate the common numerical construction. Equality for the original data requires additional horizon assumptions and a separate variational argument.

1.2 From time symmetry to spacetime data

When $K = 0$, the dominant energy condition becomes $R_g \geq 0$ and a marginal boundary is minimal. Geroch introduced the connected-surface Hawking-mass monotonicity underlying smooth inverse mean curvature flow [21]. Jang and Wald related this method to the Penrose bound under the assumption that a classical flow runs from the inner boundary to asymptotically round spheres [27]; Huisken and Ilmanen recount these precursors in [25, p. 359]. Two different flow methods then established the three-dimensional Riemannian inequality. Huisken and Ilmanen’s weak inverse mean curvature flow allows jumps while preserving the Hawking mass comparison; its bound applies to each connected component of an outermost minimal boundary [25]. Bray’s conformal flow changes the metric, preserving the horizon area and making mass nonincreasing, and proves the sharp bound for the total area of a possibly disconnected horizon [7]. Bray and Lee extended that method through dimension seven [10]. Their numerical theorem requires no spin assumption; the additional spin hypothesis in the published statement belongs to its equality conclusion.

Above dimension seven, minimizing frontiers may be singular. Bi and Zhu prove the full-area Riemannian inequality in all dimensions for almost-minimizing Caccioppoli boundaries with minimal regular part that are their own outermost minimizing enclosures, assuming a $C^{2,\alpha}$ metric extension across the boundary [6, Theorem 1.2]. Their conformal flow already treats singular enclosures. The companion [38] reconstructs the numerical argument for the locally perimeter-minimizing frontiers with smooth ambient extensions arising here; Section 2 quotes its precise statement.

Nonzero K introduces a different difficulty: neither scalar curvature nor the boundary mean curvature has the required Riemannian sign. Jang introduced a graph equation for extending Geroch’s positive-energy argument beyond time symmetry [26]. Schoen and Yau’s minimal-surface argument established the three-dimensional Riemannian positive-mass theorem [41]. Their spacetime proof combined Jang’s equation with conformal deformation and a treatment of graph blow-up to prove positive energy for general three-dimensional initial data [42]. The Penrose problem asks more of the deformation: a lower bound for enclosing area must survive along with the mass comparison. Malec and Ó Murchadha explained the obstruction to obtaining both controls from the direct classical-Jang and conformal-deformation scheme, even in spherical symmetry [33].

In spherical symmetry, Malec and Ó Murchadha proved the inequality for maximal data, and Hayward obtained it from monotonicity of the Misner–Sharp energy in untrapped regions without that maximality restriction [32, 24]. Bray and Khuri introduced a warped graph and a generalized Schoen–Yau scalar-curvature identity [8, 9]. On solutions of the generalized Jang equation, the identity separates the nonnegative dominant-energy contribution and square terms from a divergence. It does not by itself solve the equations needed to control that divergence.

They gave a new proof in spherical symmetry and proposed coupled systems for the general problem. Han and Khuri established existence and boundary blow-up for prescribed warping factors with suitable boundary and asymptotic behavior [23]. Jaracz later constructed spherical data for which one particular coupling, generalized Jang with zero divergence, has no smooth radial solution with the required positive warp and asymptotics [28]. That obstruction concerns the specified coupling and radial solution class.

Sharp spacetime results are also known beyond the spherical three-dimensional case. Bryden, Khuri and Sormani proved the sharp inequality and Schwarzschild rigidity in spherical symmetry in every spatial dimension [12]. Khuri and Kunduri proved a sharp spacetime inequality for $SU(\ell + 1)$ -invariant data of dimension $2(\ell + 1)$; their result for the full cohomogeneity-one class is Riemannian [30].

For nonsymmetric three-dimensional asymptotically flat data, Allen, Bryden, Kazaras and Khuri proved a universal invariant-mass bound with a suboptimal constant under their asymptotic hypotheses, including extra decay of $\text{tr}_g K$ [2]. For each end, their statement uses at least one outermost collection of future or past marginal components and its least enclosing area. Theorems 1.3 and 1.5 give the sharp coefficient under the original-data hypotheses above, using enclosing hypersurfaces for any weakly future outer trapped boundary.

Other nonsymmetric results concern different hypotheses and area quantities. Ellithy's inequality bounds the total area of outermost future apparent-horizon sections under a quasi-final-state condition on the future development, negative ingoing expansion, and a piecewise smooth horizon tube with finitely many area-nondecreasing jumps [20, Theorem 4.12]. Dong's pure-trace theorem gives the bound for each connected generalized-horizon component [19, Theorem 1.2]; his two-convex theorem assumes a connected outermost past apparent horizon [18, Theorem 1.2]. In the latter class the stated asymptotic restrictions force zero ADM linear momentum, as noted in [18, Remark 1.3]. These area and tensor assumptions are distinct from the enclosing hypersurfaces and arbitrary second fundamental forms in our statements. Mars surveys the underlying alternatives in formulating the initial-data Penrose problem [34].

1.3 How the proof controls curvature, mass and area

Bray–Khuri's warped scalar-curvature identity gives a route to the Riemannian inequality, but the coupled equations must also control the area changed by conformal deformation [9, Sections 6–7]. For the strict, zero-momentum data of Definition 2.1, we solve modified trace and divergence equations on a smooth filling, with a penalty that keeps the conformal exponent $t > -\varepsilon$ for any prescribed small $\varepsilon > 0$, protecting every enclosing area. The penalty enters with a favorable curvature sign, while its support lies where the warp is exponentially small; this makes its possible ADM energy cost tend to zero despite its pointwise strength. To solve this coupling, we bound the combined graph and conformal variable by testing the divergence equation on the graph before uniform ellipticity is known; together with the lower barrier, this controls both equations and closes the existence argument in Section 4.

The filling creates a separate geometric difficulty: a minimizing frontier could enter the added region, where the original energy condition gives no curvature control. We use the graph height to place a large conformal factor around that region, with transition derivatives small enough to retain nonnegative scalar curvature on the exterior height band. The factor leaves the ADM energy unchanged, and density bounds on both sides of a possible obstacle contact force its perimeter above that of a distant competitor. Thus the minimizing frontier lies in the original exterior, where its full area inherits the lower metric bound (Section 5).

The coupled system requires compactly supported K , strict energy and trapping inequalities, and end decay at every fixed derivative order. Allen–Bryden–Kazaras–Khuri already combine density with a vacuum-end deformation to obtain invariant mass while preserving least enclosing area in dimension three [2, Section 5]; here we must reach the specified input class in every dimension and compare all original enclosing cuts.

We join the original end to a vacuum slice with the same ADM energy and momentum, allowing nonvacuum original data and imposing no parity condition. Matching these charges controls the constant moments of the leading constraint error; integrability makes its first moments sublinear. Compactly supported corrections then leave a deficit in the dominant energy condition that a conformal change repairs at vanishing energy cost. We bend the vacuum slice to rest, so the resulting energy approaches $\sqrt{E^2 - |P|^2}$. A proper comparison map sends each enclosing cut of the new data to an original cut with at most a vanishing relative area increase, giving the required lower bound for the new enclosing infimum.

For each fixed prepared datum with metric g_0 and energy E_0 , these arguments produce Riemannian exteriors with nonnegative scalar curvature, energy at most $E_0 + o(1)$, and full frontier area at least $e^{-(n-1)\varepsilon} A_{\min, g_0}(S)$. The Riemannian endpoint is supplied by Bray–Lee in dimensions three through seven [10, Theorem 1.4], and by the minimizing-frontier companion theorem in all dimensions [38, Theorem 1.1]. Apply that theorem at each finite deformation parameter, first let the energy error tend to zero, and then let $\varepsilon \downarrow 0$, before removing the end preparation. Section 2 gives this numerical deduction, including the deduction of future timelikeness.

Sections 3–5 prove the common deformation; the end preparations follow in Sections 6–8. Appendices A and B contain the scalar identity calculation and local regularity estimates. The boundary-graph companion [37] develops alternative analytic constructions, and the rigidity companion [39] studies equality on the original data. Neither is a premise of the numerical proof here.

2 The geometric comparison

The central construction produces a Riemannian exterior whose energy is almost no larger than the input energy and whose full boundary area is almost no smaller than the input’s least enclosing area. We state its input hypotheses and output estimates, then apply the Riemannian Penrose inequality. Sections 3–5 prove the construction for any data satisfying these hypotheses, regardless of how they were obtained.

Definition 2.1 (Input hypotheses for the deformation). Let (Ω^n, g_0, K_0) be a smooth connected orientable one-ended exterior with nonempty compact smooth boundary S , complete with S included and with compact complement of its Euclidean coordinate end. Use the constraint and ADM conventions of Equations (1.1)–(1.3). Let ϱ be a smooth positive function on the closed exterior, equal to the coordinate radius on a distant tail. Require that, for some $0 < \beta < 1$ and $c > 0$,

$$\begin{aligned} \text{supp } K_0 \text{ is compact,} \quad g_0 - \delta = O_d(r^{2-n}) \text{ for every fixed } d, \\ R_{g_0} = O(r^{-n-\beta}), \quad \mu_0 - |J_0|_{g_0} \geq c\varrho^{-n-\beta}, \quad H_S + \text{tr}_S K_0 < 0 \text{ on every component of } S. \end{aligned} \tag{2.1}$$

The constraint densities are integrable. The ADM momentum is $P_0 = 0$; write E_0 for the energy. Finally the enclosing-area infimum, with every component and all contact counted as in

Definition 1.2, is $A_0 = A_{\min, g_0}(S) > 0$.

Once one datum satisfying Definition 2.1 has been fixed, we suppress its subscript and write g, K, E, a for g_0, K_0, E_0, A_0 . All constants in its deformation may depend on this fixed datum. End preparation will also provide proper maps comparing the prepared metric with the original metric; the deformation requires only Definition 2.1.

The output below may have a singular frontier. For the closure of a connected open exterior whose compact frontier is the support of the perimeter measure of its filled side, length distance is computed by rectifiable paths in the closed exterior and may equal $+\infty$. Completeness means completeness on each finite-distance equivalence class; every frontier point and the full perimeter remain included.

Theorem 2.2 (The geometric deformation). *Fix an exterior satisfying Definition 2.1 and $0 < \epsilon < 1$. For all sufficiently large integers N , there is a smooth Riemannian metric $\tilde{g}_N$ on an ambient manifold and a connected closed exterior X_N in that manifold. Its compact frontier is $\Sigma_N = \partial X_N$, and X_N is complete with this frontier included in the preceding length-distance convention. The induced metric on X_N has one Euclidean coordinate end with compact complement, and*

$$\begin{aligned} R_{\tilde{g}_N} &\geq 0, \quad R_{\tilde{g}_N} \in L^1(X_N), \quad \tilde{g}_N - \delta = O_2(r^{2-n}), \\ R_{\tilde{g}_N} &= O(r^{-n-\beta'}) \quad \text{for some } \beta' > 0, \\ E(\tilde{g}_N) &\leq E_0 + P_N(e^{-N\epsilon} + e^{-N\epsilon/2}), \\ \mathcal{H}_{\tilde{g}_N}^{n-1}(\Sigma_N) &\geq e^{-(n-1)\epsilon} A_0. \end{aligned} \tag{2.2}$$

Here P_N is a polynomial in N , with coefficients depending on the fixed datum and ϵ . The metric is smooth through the frontier. The whole frontier is outer minimizing and locally perimeter minimizing as the boundary of its filled side. It is smooth embedded minimal away from a compact singular set of Hausdorff dimension at most $n - 8$; for $3 \leq n \leq 7$ the singular set is empty. Its area counts every component.

The proof is completed in Section 5. The analytic construction in Sections 3 and 4 takes place on a smooth filling and yields the energy estimate before any frontier is selected. The height argument then gives a detached minimizing frontier and the lower bound for its full area.

Here is the precise Riemannian input from the complete companion *Conformal flow and the Riemannian Penrose inequality with minimizing frontiers* [38]. We use its Theorem 1.1 in the following form.

Let X^n be a connected complete closed exterior in the sense just specified, including its compact frontier Σ , with a metric γ smooth through that frontier. Suppose there is one Euclidean coordinate end with compact complement, $\gamma - \delta = O_2(r^{-\tau})$ with $\tau > (n-2)/2$, and $R_\gamma \geq 0$, $R_\gamma \in L^1$, and $R_\gamma = O(r^{-n-\beta})$ for some $\beta > 0$. Suppose its full frontier is outer minimizing and locally perimeter minimizing as the boundary of the filled side, allowing compactly supported variations on either side. Its regular part is a smooth embedded minimal hypersurface. Its singular set has Hausdorff dimension at most $n - 8$. Then

$$E(\gamma) \geq \frac{1}{2} \left(\frac{\mathcal{H}_\gamma^{n-1}(\Sigma)}{\omega_{n-1}} \right)^{(n-2)/(n-1)}.$$

The assertion includes zero frontier area and imposes neither a spin condition nor connectedness of the frontier.

The smooth numerical theorem of Bray–Lee [10, Theorem 1.4] is an alternative in dimensions three through seven. In higher dimensions the companion retains the full singular-frontier argument. Its conformal-flow ancestry includes Bi–Zhu’s version 2 proof that area remains constant before establishing nestedness, together with its separation argument for singular boundaries [6]. The independent smooth positive-mass input is Brendle–Wang [11, Corollary 1.6], applied there to constructed smooth complete boundaryless approximants with strictly positive scalar curvature and the required differentiated end. Those stronger approximant hypotheses are not additional assumptions on X .

Theorem 2.3 (Energy and enclosing area for the deformation inputs). *Every exterior in Definition 2.1 satisfies*

$$E_0 \geq \frac{1}{2} \left(\frac{A_0}{\omega} \right)^{(n-2)/(n-1)}. \quad (2.3)$$

Proof. Fix the datum and ϵ . Apply the quoted Riemannian theorem to the actual exterior X_N furnished by Theorem 2.2. Its metric is smooth through the compact frontier; its closure is complete and its open exterior is connected. The end has compact complement and decay exponent $n - 2 > (n - 2)/2$. Its scalar curvature is nonnegative and integrable with the required pointwise decay. The whole frontier is outer minimizing and locally perimeter minimizing, hence has minimal regular part and singular dimension at most $n - 8$. These verify each receiving hypothesis. We obtain

$$E_0 + P_N(e^{-N\epsilon} + e^{-N\epsilon/2}) \geq E(\tilde{g}_N) \geq \frac{1}{2} \left(\frac{\mathcal{H}_{g_N}^{n-1}(\Sigma_N)}{\omega} \right)^{(n-2)/(n-1)} \geq \frac{1}{2} \left(\frac{e^{-(n-1)\epsilon} A_0}{\omega} \right)^{(n-2)/(n-1)}.$$

First let $N \rightarrow \infty$; exponential decay beats the fixed polynomial. Then let $\epsilon \downarrow 0$. These are limits of numbers. The exhaustion constructing a smooth solution at each fixed N is completed before this argument, and no metric at $N = \infty$ is required. Since $A_0 > 0$, this also proves $E_0 > 0$. $\square$

Deduction of Theorem 1.3. Proposition 6.1, proved in Section 6, supplies data satisfying Definition 2.1 with energies $E_j \rightarrow m = \sqrt{E^2 - |P|^2}$ and enclosing infima $A_j \geq (1 - o(1))A_{\min,g}(S)$. Apply Theorem 2.3 to each of these data sets, completing its fixed-data limits before varying j . Then

$$m \geq \frac{1}{2} \left(\frac{A_{\min,g}(S)}{\omega} \right)^{(n-2)/(n-1)}.$$

No estimate uniform in the preparation index is used.

For sharpness, let $M > 0$, $r_M = (2M)^{1/(n-2)}$, and take the static Schwarzschild–Tangherlini slice

$$g = \frac{dr^2}{1 - 2M/r^{n-2}} + r^2 \sigma_{n-1}, \quad K = 0, \quad r \geq r_M.$$

Here σ_{n-1} is the unit round metric on $\mathbb{S}^{n-1}$. Proper distance from the horizon makes this a smooth complete metric with minimal compact boundary. Its scalar curvature is zero, and its ADM vector is $(M, 0)$. Radial projection to the sphere of radius r_M has $(n - 1)$ -Jacobian at most one. More explicitly the pullback of that sphere’s area form is closed with comass at most one; Stokes’ theorem on an end truncation of any full outer domain shows that its integral over the

entire cut is ωr_M^{n-1} . Thus every enclosing hypersurface has area at least that number, while the boundary itself attains it. Consequently $A_{\min} = \omega(2M)^{(n-1)/(n-2)}$ and equality holds. $\square$

The same numerical bound forces future timelikeness once a positive enclosing area is retained under an energy-raising conformal change.

Proof of Corollary 1.4. If instead $E \leq |P|_\delta$, choose $\varepsilon_j > 0$ tending to zero and put

$$\lambda_j = \frac{|P|_\delta + \varepsilon_j - E}{2} > 0.$$

Lemma 6.6, proved in Section 6.6, gives, separately for each j , data satisfying the same one-ended strong-decay, constraint and trapping hypotheses, with

$$E_j = |P|_\delta + \varepsilon_j, \quad P_j = P, \quad A_{\min, g_j}(S) \geq A_{\min, g}(S) > 0.$$

Their ADM vectors are future timelike, so Theorem 1.3 gives

$$\sqrt{(|P|_\delta + \varepsilon_j)^2 - |P|^2_\delta} \geq \frac{1}{2} \left(\frac{A_{\min, g}(S)}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} > 0.$$

The right side is fixed and the left side tends to zero, a contradiction. The theorem is applied afresh to each datum; no estimate for its end preparation is required to be uniform near the null cone. Thus $E > |P|_\delta$, and Theorem 1.3 applies to the original data. $\square$

The weak-decay branches use different end reductions. In dimension three, the enclosing-area infima converge; in dimension four, the rest-end replacement supplies the one-sided comparison needed by the numerical bound. The four-dimensional proof then makes the inequalities strict at each fixed replacement radius and removes that perturbation before sending the radius to infinity.

Deduction of Theorem 1.5. In dimension three, the enclosing cuts in Definition 7.1 are those in Definition 1.2. Dividing the geometric constraint densities by 8π gives the physical densities used in Section 7, preserving the dominant energy condition and integrability. The ADM normalization is unchanged. Lemma 7.3 gives positive enclosing area, and Theorem 7.6 gives future timelikeness and Equation (1.5), since $\omega_2 = 4\pi$.

In dimension four, the one-ended case of Definition 8.1 is exactly Definition 1.2, and both the constraint and ADM normalizations agree with those of Definition 1.1. Lemma 8.3 gives positive enclosing area. The one-ended case of Theorem 8.2 then gives future timelikeness and the equivalent mass bound (8.4), which is Equation (1.5) with $n = 4$. $\square$

3 A scalar identity that controls area and mass

Throughout this section the data satisfy Definition 2.1. Fix $0 < \epsilon < 1$. The unknowns f, Z, t are real-valued smooth functions on the filled manifold constructed below. The positive function $l : \mathbb{R} \rightarrow (0, \infty)$ and the positive number η are fixed by Equation (3.1). We seek a metric $\hat{g} = e^{2t}(g + l(t)^2 df \otimes df)$ with $t > -\epsilon$, so that $\hat{g} \geq e^{-2\epsilon}g$ controls the area loss for every enclosing hypersurface. We solve for the graph height f and the modified conformal variable Z , with t determined by Equation (3.2). The normalized height $h = \eta f$ will separate the filling from the enclosure. A weighted quadratic estimate controls its derivatives, and the divergence equation for Z controls the penalty's contribution to the ADM energy. Section 4 constructs smooth solutions by exhaustion at fixed ϵ, N ; the later limit in N uses only the explicit estimates. All identities below hold in every dimension $n = k + 1 \geq 3$.

3.1 Filling the boundary and choosing the unknowns

We need a divergence equation with no inner flux. A smooth filling permits that equation to be solved on a complete manifold without boundary. The filling is auxiliary: its energy condition will never be assumed. Its height will later be separated from the Riemannian comparison region.

The original exterior has compact complement of its coordinate end. Cut it at a large coordinate sphere, take an orientation-reversed copy of the resulting compact manifold, and cap the copied spherical face by a ball. The resulting compact manifold has boundary a copy of the *whole* S . Gluing along that boundary therefore fills all components of S simultaneously; no individual component is required to bound. Smooth collars perform the gluing. Initially extend g, K across S on a fixed short collar. Strict negativity of the expansions of the signed-distance leaves persists there by continuity. Beyond that collar interpolate the metric to a product, and subtract a smooth nonnegative multiple of the metric from K . Start this subtraction while the strict inequality still holds and make the multiple large before performing the remaining interpolation. Since the transition is fixed and compact, its leaf mean curvatures and the other tangential traces are bounded. Thus the subtraction can ensure

$$H + \text{tr}_{\text{leaf}} K < -c_0 < 0$$

through every transition. Arrange $K = -L_0 g$ on the ensuing product necks and on the remaining cap, with L_0 a sufficiently large fixed constant. The neck length at each boundary component may be chosen between $c(1 + N)$ and $C(1 + N)$, increasing C when the height ramp in Section 5 is chosen.

Denote the resulting complete manifold without boundary by $\mathcal{M}_N$, and continue to write g, K for the extended fields. They are unchanged on the original exterior. All the local metric models belong to a fixed finite family, apart from translation along product cylinders. Consequently their injectivity radii are bounded below, every prescribed finite collection of local geometric bounds is bounded independently of N , and the enlarged core has volume and covering number $O(1 + N)$. No energy condition is imposed in the added region. Extend the end radius to a smooth positive function ϱ , uniformly comparable to 1 on that core. Hence weighted integrals over the core cost at most a polynomial in N .

In the scalar construction, $p = p(t)$ is the cutoff defined below; the dimensional exponent is written $n - 2 = k - 1$. With ϵ fixed as above, let $\vartheta \in C^\infty(\mathbb{R})$ be nonincreasing, equal to 1 on $(-\infty, 0]$ and to 0 on $[1, \infty)$, with values in $[0, 1]$. For an integer $N > \max(k, 2/\epsilon)$ put

$$\begin{aligned} p(t) &= N\vartheta(Nt), & l(t) &= \exp\left(\int_0^t p(s) ds\right), \\ \ell &= e^{-N\epsilon}, & \eta &= \ell^{3/2}, & m_0(t) &= \vartheta(N(t + \epsilon)). \end{aligned} \tag{3.1}$$

In particular $l = e^{Nt}$ for $t \leq 0$, $l \leq e$ everywhere, and $\sup |p^{(j)}| \leq C_j N^{j+1}$ for each fixed j .

For smooth unknowns f, Z , define t implicitly by

$$Z = t + \frac{1}{2k} \log D, \quad D = 1 + l(t)^2 |df|_g^2. \tag{3.2}$$

For fixed df the derivative of the right side with respect to t is $(k + pv)/k \geq 1$, where $v = 1 - D^{-1}$. Its limits as t tends to $\pm\infty$ are $\pm\infty$. Thus this equation defines t globally and smoothly as a function of $(Z, |df|^2)$, without a lower-bound assumption on t . Use g to identify vectors and

covectors and set

$$\begin{aligned}
w &= lD^{-1/2}\nabla f, & a_w &= |w|, & v &= a_w^2, & d &= D^{-1} = 1 - v, \\
\chi &= \frac{kd}{k+pv}, & A_d &= I - w \otimes w, & A_\chi &= I - \frac{k+p}{k+pv}w \otimes w, \\
H^f &= lD^{-1/2}\text{Hess}_g f, & L &= K + H^f, & F &= \text{tr}_{A_\chi} L, & h &= \eta f, \\
u &= e^{(k-1)t}l\sqrt{D}, & V &= uA_\chi(k\nabla Z + K(w, \cdot)), \\
\hat{g} &= e^{2t}(g + l^2 df \otimes df), & s_p &= p + \frac{k-1}{2}.
\end{aligned} \tag{3.3}$$

For a symmetric contravariant two-tensor A and a covariant two-tensor B , write $\text{tr}_A B = A : B = A^{ij}B_{ij}$. We use g to raise indices when the displayed A is an endomorphism. For covectors α, β , write $\langle \alpha, \beta \rangle_A = A^{ij}\alpha_i\beta_j$ and $|\alpha|_A^2 = \langle \alpha, \alpha \rangle_A$; for a covariant two-tensor, $|B|_{A \otimes A}^2 = A^{ia}A^{jb}B_{ij}B_{ab}$. The notation $B^\sharp$ means the endomorphism obtained by raising its first index with g ; we write Bw for $B^\sharp(w)$. For a scalar F , $w(F) = dF(w)$. The tensors A_d, A_χ have transverse eigenvalue 1 and axial eigenvalues d, χ , respectively. More explicitly,

$$0 < \frac{k}{k+N}d \leq \chi \leq d \leq 1, \quad k dZ = (k+pv) dt + H^f(w, \cdot). \tag{3.4}$$

The second equality follows by differentiating Equation (3.2): $\frac{1}{2}d \log D = pv dt + H^f(w, \cdot)$. Every tensor in Equation (3.3) is smooth at $df = 0$. The axis notation used below is only a way of displaying these tensors, not a coefficient definition at gradient zeros.

In this and the next two sections $\mathcal{P}_N$ denotes a polynomial in $1 + N$, whose coefficients may depend on the fixed input data, n , and ϵ . Increasing such a polynomial finitely many times does not change this convention. An estimate with an arbitrary constant depending on fixed N will be explicitly identified as such; it cannot be substituted into a later $\mathcal{P}_N$ estimate.

3.2 The scalar identity

The model for this calculation is the Schoen–Yau scalar-curvature identity for Jang graphs [42], and more specifically its warped generalization in [9, Identity 9, p. 580]. Here the trace is taken against A_χ , whose distinguished direction is ∇f ; the conformal variable and the divergence are also changed. We derive their exact identity before imposing the coupled equations; its algebra and their global solvability are separate steps.

At a nonzero gradient let $e = \nabla f/|df|$, and decompose L and dt relative to $e^\perp \oplus \mathbb{R}e$ as

$$L = \begin{pmatrix} T & M \\ M^\top & j \end{pmatrix}, \quad dt = (y, x), \quad T^0 = T - \frac{\text{tr } T}{k}I_{e^\perp}.$$

Here $\text{tr } T = F - \chi j$. At $df = 0$ one may choose any unit axis.

Proposition 3.1 (Scalar identity). *Let g be any smooth Riemannian metric in dimension $n = k + 1 \geq 3$, let K be any smooth symmetric covariant two-tensor, and let f, Z be smooth real functions. With the parameters and tensors in Equations (3.1)–(3.3), the metric $\hat{g}$ satisfies*

$$\frac{1}{2}e^{2t}R_{\hat{g}} = \mu + J(w) + \mathcal{T} + w(F) - F\tau - u^{-1}\text{div}_g V, \tag{3.5}$$

where $\tau = \text{tr}_g K$, μ, J are defined by Equation (1.1), and

$$\begin{aligned} \mathcal{T} = & \frac{1}{2}|T^0|^2 + |M + s_p a_w y|^2 + \left(ks_p - \frac{(k-1)^2}{4}v\right)|y|^2 + ks_p dx^2 \\ & - \frac{k-1}{2k}(F - \chi j)^2 + \chi j^2 - \chi F j + F^2 + 2s_p \chi j a_w x. \end{aligned} \quad (3.6)$$

The quantity $\mathcal{T}$ is smooth, including where $df = 0$.

The identity follows by viewing $g + l^2 df \otimes df$ as the spatial metric in the stationary Lorentz metric $-l^2(dz - df)^2 + (g + l^2 df \otimes df)$, and then applying the conformal scalar-curvature law. The change from t to Z places the second conformal derivatives in $u^{-1} \text{div } V$; the remaining first-order terms form $\mathcal{T}$. The complete calculation, including the terms containing p' , is in Appendix A. The next proposition identifies the positive quantities retained by this rearrangement.

3.3 Coercivity with polynomial constants

The identity has isolated the divergence, but it is useful only if the remaining quadratic form controls the graph and conformal derivatives. The next proposition records exactly which axial weights survive. Its constants are polynomial in the large parameter; this is what will allow the exponentially small penalty flux to tend to zero.

Proposition 3.2 (Weighted coercivity). *For $0 \leq p \leq N$ and $0 \leq v < 1$ the quadratic form in Equation (3.6) is nonnegative, and*

$$|T|^2 + |M|^2 + dj^2 + F^2 + |dt|_{A_d}^2 \leq C_k(1 + N)^2 \mathcal{T}. \quad (3.7)$$

At $dt = 0$ one has, with constants depending only on k ,

$$c_k(|T^0|^2 + |M|^2 + F^2 + \chi j^2) \leq \mathcal{T} \leq C_k(|T^0|^2 + |M|^2 + F^2 + \chi j^2). \quad (3.8)$$

Proof. The entire axial part of Equation (3.6) is exactly

$$\begin{aligned} & ks_p d \left(x + \frac{a_w j}{k + pv} \right)^2 + \frac{k+1}{2k} \left(F - \frac{\chi j}{k+1} \right)^2 + \frac{k dB_k(v)}{(k + pv)^2} j^2, \\ & B_k(v) = d \frac{k(k+2)}{2(k+1)} + v \frac{k+1}{2} = \frac{k(k+2) + v}{2(k+1)}. \end{aligned} \quad (3.9)$$

Indeed completing first the x square and then the F square leaves

$$\chi - \frac{k}{2(k+1)} \chi^2 - s_p \frac{\chi^2 v}{kd} = \frac{k dB_k(v)}{(k + pv)^2}.$$

The last coefficient is at least $c_k d / (1 + N)^2$, and

$$ks_p - \frac{(k-1)^2}{4}v \geq \frac{k^2 - 1}{4} > 0.$$

The completed squares control the shifted variables; we must recover the original variables without multiplying two losses in N . Let $c_j = k dB_k(v) / (k + pv)^2$ be the last coefficient in Equation (3.9). Its ratios satisfy

$$\frac{d a_w^2 / (k + pv)^2}{c_j} = \frac{v}{k B_k(v)} \leq C_k, \quad \frac{\chi^2}{c_j} = \frac{kd}{B_k(v)} \leq C_k.$$

Thus the error in recovering $\sqrt{d}x$ from its shifted square, and the error in recovering F from its shifted square, are each controlled directly by the residual $c_j j^2$. The bound $c_j \geq c_k d / (1 + N)^2$ is needed only to recover dj^2 itself. The transverse coefficient above controls $|y|^2$, so recovering M from $M + s_p a_w y$ costs at most $C_k(1 + N)^2$. Finally $\text{tr } T = F - \chi j$ and $|T|^2 = |T^0|^2 + (\text{tr } T)^2/k$ control $|T|^2$. These estimates prove Equation (3.7) with one quadratic polynomial, while retaining the axial factors d .

When $dt = 0$ the axial expression instead completes as

$$\frac{k+1}{2k} \left(F - \frac{\chi j}{k+1} \right)^2 + \chi \left(1 - \frac{k\chi}{2(k+1)} \right) j^2.$$

Its second parenthesis is bounded above and below by positive constants, and $\chi^2 \leq \chi$. This proves Equation (3.8) independently of N . $\square$

3.4 The penalized system and the lower barrier

We now choose the equations using the signs in Equation (3.5). Imposing $F = h = \eta f$ turns $w(F)$ into the nonnegative term $\eta a_w |df|$. We can assign a small fixed fraction of this term and of $\mathcal{T}$ to the divergence source while retaining the rest in the scalar-curvature formula. To enforce the lower bound for t , the source will also contain a negative term near $t = -\epsilon$. Its sign improves scalar curvature, but its integral can increase ADM energy. We therefore need both a pointwise floor argument and a separate estimate for the weighted integral of that penalty.

Choose

$$\frac{n}{4} < b_1 < \min \left(n - 2, \frac{n}{2} \right), \quad \rho_0 = \varrho^{-n-\beta}, \quad \rho_1 = \varrho^{-2b_1}, \quad \rho = c_* \rho_0 > 0. \quad (3.10)$$

This interval is nonempty for every $n \geq 3$. Its three restrictions serve different parts of the argument: $b_1 < n - 2$ permits the radial height comparison in Section 4; $2b_1 < n$ lets $v\rho_1$ dominate the end-curvature error in the floor calculation; and $4b_1 > n$ makes ρ_1^2 integrable in the flux estimate. Take c_* small enough that $\mu - |J| \geq 2\rho$ on the original exterior. The system to be solved is

$$\begin{aligned} F = h = \eta f, \quad \text{div } V = u\Xi, \\ \Xi = \delta_0(\mathcal{T} + \eta a_w |df|) + \rho - m_0 C_N(\rho_0 + v\rho_1). \end{aligned} \quad (3.11)$$

Here $\delta_0 > 0$ is a sufficiently small fixed number and C_N is a sufficiently large polynomial, specified by the next lemma. Substitution in the scalar identity retains $(1 - \delta_0)(\mathcal{T} + \eta a_w |df|)$; the positive source ρ spends only part of the strict energy margin on the original exterior. The remaining term $-h\tau$ will be controlled on a small height band. The negative penalty is chosen to contradict the flux lower bound at a minimum with $t = -\epsilon$.

For a solution with $t > -\epsilon$, the penalty is supported in $-\epsilon < t < -\epsilon + N^{-1}$, where $\ell \leq l \leq \ell\ell$. Proposition 3.4 will use this small weight to bound the possible energy increase by a polynomial times $\ell + \ell^2/\eta$. The height-derivative estimate below instead uses η/ℓ . The choice $\eta = \ell^{3/2}$ makes both ratios ℓ^2/η and η/ℓ equal to $\ell^{1/2}$. Both ratios tend to zero faster than any polynomial in N at fixed ϵ .

On a large truncation the outer data are $f = Z = 0$; there is no inner boundary. For the existence argument we use two homotopies. In the first replace K by aK , $0 \leq a \leq 1$, everywhere in these definitions, keeping the weights and constants fixed. In the second keep $K = 0$ and multiply the whole expression Ξ by $s \in [0, 1]$.

Lemma 3.3 (Pointwise exclusion of the floor). *For sufficiently small $\delta_0 > 0$, depending only on k , one can choose a polynomial C_N with the following property. Consider a smooth solution of either homotopy specified above on a truncation, with outer data $f = Z = 0$. Suppose its height satisfies $|h| \leq C$ on the core and $|h| \leq C\varrho^{-b_1}$ on the end, with C independent of N and the homotopy parameter. If $\min t = -\epsilon$, no interior point attains that minimum. The polynomial may depend on this height bound and the fixed data, as in the convention for $\mathcal{P}_N$.*

Proof. For a symmetric tensor B write

$$\mathcal{S}(B) = \frac{1}{2} \left\{ (\text{tr}_{A_d} B)^2 - |B|_{A_d \otimes A_d}^2 \right\}.$$

We first prove that, for a constant $c_1 > 0$ depending only on k ,

$$\mathcal{T} - \mathcal{S}(H^f) \geq c_1 \mathcal{T} - \mathcal{P}_N(vF^2 + |K|^2) \quad \text{when } dt = 0. \quad (3.12)$$

At $dt = 0$, direct block contraction gives

$$\mathcal{S}(L) = \frac{k-1}{2k} (F - \chi j)^2 - \frac{1}{2} |T^0|^2 + dj(F - \chi j) - d|M|^2.$$

Subtracting from Equation (3.6) gives the full difference, including its mixed term:

$$\begin{aligned} \mathcal{T} - \mathcal{S}(L) &= |T^0|^2 + (1+d)|M|^2 + \frac{F^2}{k} \\ &\quad + \left(\frac{k-2}{k} \chi - d \right) Fj + \chi \left(1 + d - \frac{k-1}{k} \chi \right) j^2. \end{aligned} \quad (3.13)$$

If $(1+p)v \leq \zeta_k$ with $\zeta_k > 0$ sufficiently small, then $|d-1| + |\chi-1| \leq C_k \zeta_k$. At $d = \chi = 1$ the expression is

$$|T^0|^2 + 2|M|^2 + \frac{(F-j)^2}{k} + j^2 = |L|^2.$$

Perturbation of this finite-dimensional positive form, and Equation (3.8), therefore bound Equation (3.13) below by a fixed positive multiple of $\mathcal{T}$.

In the other case $1/v \leq (1+p)/\zeta_k$. The last coefficient in Equation (3.13) is at least χ , since $\chi \leq d$. The absolute value of its mixed coefficient is at most d . Hence Young's inequality bounds the difference below by

$$|T^0|^2 + |M|^2 + \frac{1}{2} \chi j^2 + \left(\frac{1}{k} - \frac{d^2}{2\chi} \right) F^2.$$

Here

$$\frac{d^2}{\chi} = \frac{d(k+pv)}{k} \leq 1 + \frac{N}{k}.$$

It follows, again using Equation (3.8), that the desired fixed multiple of $\mathcal{T}$ is obtained after subtracting at most $C_k(1+N)^2 v F^2 / \zeta_k$.

To replace L by $H^f = L - K$, polarize $\mathcal{S}$ in the $A_d \otimes A_d$ norm. The preceding stationary coercivity gives

$$|L|_{A_d \otimes A_d}^2 = |T|^2 + 2d|M|^2 + d^2 j^2 \leq C_k(1+N)\mathcal{T}.$$

Moreover $|K|_{A_d \otimes A_d} \leq |K|$. Therefore

$$|\mathcal{S}(L) - \mathcal{S}(L - K)| \leq \zeta \mathcal{T} + C_{k,\zeta}(1+N)|K|^2$$

for every fixed $\zeta > 0$. Taking ζ below half the positive constant already obtained proves Equation (3.12).

At an interior minimum of t , $dt = 0$ and $\text{Hess}_g t \geq 0$. To estimate curvature from above, view $\bar{g} = g + l^2 df \otimes df$ as the metric on the graph of f in the Riemannian warped product $g + l^2 dz^2$. At that point $dl = 0$ and $l^{-1} \text{Hess} l = p \text{Hess} t$. Its ambient scalar curvature is $R_g - 2p\Delta t$, its horizontal Ricci tensor is $\text{Ric}_g - p \text{Hess} t$, and its unit vertical Ricci component is $-p\Delta t$. The graph unit normal is $D^{-1/2} \partial_{z, \text{unit}} - w$ and its second form with this orientation is $-H^f$, which has the same quadratic Gauss term as H^f . The Gauss equation followed by the conformal law consequently gives the exact minimum-point formula

$$\begin{aligned} \frac{1}{2} e^{2t} R_{\hat{g}} &= \frac{1}{2} R_g - v \text{Ric}_g(e, e) + \mathcal{S}(H^f) \\ &\quad - p v \text{tr}_{e^\perp} \text{Hess} t - k \text{tr}_{A_d} \text{Hess} t \\ &\leq \frac{1}{2} R_g - v \text{Ric}_g(e, e) + \mathcal{S}(H^f). \end{aligned} \tag{3.14}$$

When $v = 0$, terms using e vanish. In particular all $p'(dt)^2$ terms vanish at this point.

Combining Equations (3.5), (3.12), and (3.14), and using $F = \eta f$, yields

$$u^{-1} \text{div} V \geq c_1 \mathcal{T} + \eta a_w |df| - \mathcal{P}_N(\rho_0 + v\rho_1). \tag{3.15}$$

Here are the weights in this estimate. The curvature-independent constraint contribution is $\mu - R_g/2 = (\tau^2 - |K|^2)/2$. This, $J(w)$, $F\tau$, and the $|K|^2$ error are uniformly bounded on the enlarged core and vanish on the far end, so they cost $C\rho_0$ pointwise. On the end, $v|\text{Ric}_g| \leq C v \varrho^{-n} \leq C v \rho_1$ because $2b_1 < n$; the height comparison gives $vF^2 \leq C v \rho_1$. The same statements hold uniformly for aK , $0 \leq a \leq 1$. No use of the dominant energy condition in the filling has entered this calculation.

At $t = -\epsilon$ one has $m_0 = 1$. Subtracting the right side of Equation (3.11) from the lower bound in Equation (3.15) gives

$$(c_1 - \delta_0) \mathcal{T} + (1 - \delta_0) \eta a_w |df| + (C_N - \mathcal{P}_N)(\rho_0 + v\rho_1) - \rho.$$

Choose $0 < \delta_0 < \min(c_1, 1)/2$ and $C_N > \mathcal{P}_N + c_* + 1$. The expression is strictly positive, contradicting the equation. In the second homotopy stage the trace comparison gives $f = 0$: a positive maximum or negative minimum contradicts $\text{tr}_{A_\chi} H^f = \eta f$. Thus $t = Z$, and at a floor minimum the left side of the divergence equation divided by u is $k\Delta t \geq 0$, whereas its right side is $s(\rho - C_N \rho_0) < 0$ for $s > 0$. At $s = 0$, the equation $\text{div}(u \nabla Z) = 0$ with zero outer data gives $Z = 0$ by the maximum principle. This finishes the pointwise exclusion. $\square$

The lemma excludes an interior minimum at $t = -\epsilon$ on the boundary of the moving degree domain. Section 4 supplies its height bounds and excludes such a minimum on the outer face. Solutions whose minima are below $-\epsilon$ are outside this argument.

3.5 Scalar curvature on solutions and height derivatives

On the original exterior, substituting Equation (3.11) into the identity gives

$$\begin{aligned} \frac{1}{2} e^{2t} R_{\hat{g}} &= \mu + J(w) - \rho - h\tau + (1 - \delta_0) \mathcal{T} + (1 - \delta_0) \eta a_w |df| \\ &\quad + m_0 C_N(\rho_0 + v\rho_1). \end{aligned} \tag{3.16}$$

In particular a small fixed height band has a strict scalar margin: choose $b_3 > 0$ such that $b_3|\tau| \leq \rho/2$ on the support of τ . Then $\mu - |J| - \rho - |h\tau| \geq \rho/2$ on $|h| \leq b_3$. This is possible since τ is compactly supported and $\mu - |J| \geq 2\rho$ there.

For later use the exact height identities are

$$\begin{aligned} |dh|_{A_d} &= \frac{\eta|df|}{\sqrt{D}} \leq \frac{\eta}{l}, \\ e^{2t}\Delta_{\hat{g}}h &= \frac{\eta}{l\sqrt{D}} \{\operatorname{div} w + (k-1-p)w(t)\}, \\ \operatorname{div} w &= \operatorname{tr}_{A_d} H^f + p dw(t). \end{aligned} \tag{3.17}$$

The middle formula follows from $\sqrt{D}A_d\nabla h = (\eta/l)w$ and the conformal Laplacian law; the last follows by differentiating $w = lD^{-1/2}\nabla f$. For a solution with $t > -\epsilon$, $l \geq \ell$, and Proposition 3.2 therefore implies

$$|dh|_{A_d} \leq \ell^{1/2}, \quad |e^{2t}\Delta_{\hat{g}}h| \leq \mathcal{P}_N \ell^{1/2}(1 + \sqrt{\mathcal{T}}). \tag{3.18}$$

Indeed $\operatorname{tr}_{A_d} H^f = \operatorname{tr} T + dj - \operatorname{tr}_{A_d} K$ is bounded by $\mathcal{P}_N(1 + \sqrt{\mathcal{T}})$. The other potentially large term is $w(t) = a_w x$. The prefactor $D^{-1/2} = \sqrt{d}$ converts it to $a_w \sqrt{d} x$, precisely the weighted quantity controlled by Equation (3.7). Thus no inverse power of d is lost. Section 5 uses these estimates to raise the metric inside a compact height band.

3.6 The penalty flux and its ADM normalization

Proposition 3.4 (Energy change). *Fix ϵ and N as above. Suppose a smooth global solution of Equation (3.11) satisfies $t > -\epsilon$, $Z \leq \mathcal{P}_N$, and on the coordinate end*

$$f = O_j(e^{-c_N r}), \quad t = O_j(r^{2-n}), \quad \Delta_g t = O(r^{-n-\beta})$$

for all required derivative orders, where $c_N > 0$ and the constants may depend arbitrarily on this fixed N . Then the ADM energy of $\hat{g}$ exists, the following flux limit and integral are finite, and

$$\begin{aligned} E(\hat{g}) - E &= -\frac{1}{k\omega} \lim_{r \rightarrow \infty} \int_{S_r} g(V, \nu_g) dA_g = -\frac{1}{k\omega} \int_{\mathcal{M}_N} u \Xi dV_g, \\ E(\hat{g}) &\leq E + \mathcal{P}_N(\ell + \ell^{1/2}). \end{aligned} \tag{3.19}$$

The polynomial in the error uses only the fixed geometry, the weights, and the explicit parameters, not the arbitrary fixed- N derivative constants in the stated end estimates.

Proof. Write $\mathfrak{F}_N$ for the flux limit in Equation (3.19), once its existence is proved. For fixed N , the end hypotheses and Equation (3.4) give

$$D - 1 = O_j(e^{-c'_N r}), \quad Z - t = O_j(e^{-c'_N r}), \quad V = k\nabla t + O_N(r^{3-2n}) + O(e^{-c'_N r}).$$

The notation O_N here permits arbitrary fixed- N constants. Since $N\epsilon > 2$ and $t \rightarrow 0$, $m_0 = 0$ sufficiently far out. There $\mathcal{T} = ks_p|dt|^2 + O(e^{-c'_N r})$, so every term in $u\Xi$ is integrable: the gradient term is $O_N(r^{2-2n})$, and $\rho = O(r^{-n-\beta})$. Smoothness gives integrability on the compact filled core. The divergence theorem on exhausting domains has no inner boundary. It equates the outer flux with the source integral and proves the flux limit exists. It also proves existence of $\lim \int_{S_r} \partial_{\nu_g} t$ directly from $\Delta_g t \in L^1$.

The graph part $l^2 df^2$ contributes zero to the ADM flux by its exponential decay. The conformal perturbation is

$$(e^{2t} - 1)g = 2t\delta + O_1(r^{4-2n}).$$

The linear ADM integrand of $2t\delta$ is $-2k \partial_i t$; the remainder has integrated flux $O(r^{2-n}) \rightarrow 0$. Euclidean and g -normal fluxes differ by the same vanishing order. Thus

$$E(\hat{g}) - E = -\omega^{-1} \lim_{r \rightarrow \infty} \int_{S_r} \partial_{\nu_g} t dA_g = -\frac{\mathfrak{F}_N}{k\omega}.$$

It remains to bound the negative part of the integral independently of those end constants. It is supported in the penalty band $-\epsilon < t < -\epsilon + 1/N$, where $t < 0$ and $\ell \leq l \leq \epsilon\ell$. Put $\sigma = |df|$. Directly from the definitions, with constants independent of N on this band,

$$u \leq C(\ell + \ell^2\sigma), \quad uv \leq C\ell^2\sigma, \quad ua_w\sigma = e^{(k-1)t}l^2\sigma^2 \geq c\ell^2\sigma^2. \quad (3.20)$$

For the second estimate use $l\sqrt{D}v = l^3\sigma^2/\sqrt{1+l^2\sigma^2} \leq l^2\sigma$. The nonnegative terms $u\delta_0\mathcal{T}$ and $u\rho$ may be dropped, while the favorable gradient term absorbs the linear σ terms. More explicitly Young's inequality gives

$$CC_N\ell^2\sigma(\rho_0 + \rho_1) \leq \frac{1}{2}\delta_0 c\eta\ell^2\sigma^2 + C'C_N^2\frac{\ell^2}{\eta}(\rho_0^2 + \rho_1^2).$$

Consequently

$$\mathfrak{F}_N \geq -CC_N\ell \int \rho_0 - C'C_N^2\frac{\ell^2}{\eta} \int (\rho_0^2 + \rho_1^2).$$

All integrals use dV_g on $\mathcal{M}_N$. They are bounded by $C(1+N)$: on the end ρ_0 is integrable and $4b_1 > n$ makes ρ_1^2 integrable, while on the core the weights are uniformly bounded and its volume is $O(1+N)$. Since $\ell^2/\eta = \ell^{1/2}$, this proves the polynomial flux estimate. For fixed $\epsilon > 0$ the resulting error tends to zero as $N \rightarrow \infty$. $\square$

Finally, every solution with $t > -\epsilon$ satisfies the pointwise comparison

$$\hat{g} \geq e^{-2\epsilon}g. \quad (3.21)$$

This comparison and the flux estimate are applied to each smooth solution at fixed N . They require no limiting elliptic solution as $N \rightarrow \infty$ and no regularity assertion for a minimizing boundary.

4 Solving the filled equations

The metric comparison and the energy estimate from Section 3 become useful only after its two scalar equations have a smooth solution. We construct that solution here, on the filled manifold and at fixed input data. The boundary of a large truncation is a single coordinate sphere; both unknowns vanish there.

The first task is to keep the conformal variable above $-\epsilon$ and to bound Z without assuming uniform ellipticity. The height comparison and the floor calculation give the lower bound. Testing the scalar divergence equation on the graph of f then gives a polynomial upper bound for Z . Together these bounds control the slope and make both scalar equations uniformly elliptic for each fixed N .

We next obtain continuity of Df from the trace equation, whose principal matrix has ∇f as an eigendirection. A second scalar estimate gives continuity of Z from its divergence equation, whose source grows quadratically in DZ . These estimates allow the regularity argument to pass between the two equations and reach smoothness. Finally we solve the trace equation for every trial Z , use it to define a compact map, and obtain a fixed point by two homotopies. Exhausting the outer spheres gives the complete solution and the end estimates needed for its ADM flux. No regularity theorem for elliptic systems or numerical Penrose inequality enters this construction.

Theorem 4.1 (Filled construction). *Fix data satisfying Definition 2.1, with its exponent $0 < \beta < 1$, and choose $0 < \epsilon < 1$ and*

$$\frac{n}{4} < b_1 < \min\{n - 2, n/2\}.$$

Use the extensions $(\mathcal{M}_N, g, K)$ constructed in Section 3, the parameters in Equation (3.1), the tensors in Equation (3.3), and the weights in Equation (3.10). Choose $\delta_0 > 0$ and the polynomial C_N as in Lemma 3.3. For all sufficiently large integers $N > k = n - 1$, there are smooth functions f, Z on $\mathcal{M}_N$ solving Equation (3.11). The associated function t satisfies

$$-\epsilon < t \leq Z \leq P(N), \quad \ell \leq l(t) \leq e, \quad |f| + |df|_g \leq \exp(P(N)), \quad \ell = e^{-N\epsilon}. \quad (4.1)$$

Here and below P is a polynomial with coefficients depending only on the fixed input data, n, ϵ , and the fixed choices in the filling. The height $h = \eta f$, $\eta = \ell^{3/2}$, obeys

$$|h| \leq C, \quad |h(x)| \leq Cr(x)^{-b_1} \quad \text{on the original end}, \quad (4.2)$$

where C is independent of N . At each fixed N , for every integer $j \geq 0$ there are constants $C_{N,j}$ and $a_N > 0$ such that on that end

$$|D^j f| \leq C_{N,j} e^{-a_N r}, \quad t, Z = O_j(r^{2-n}), \quad \Delta_g t = O(r^{-n-\beta}). \quad (4.3)$$

The constants in Equation (4.3), and higher derivative bounds on compact sets, may depend arbitrarily on fixed N .

For clarity, we specify the homotopies used in the proof. On a large outer truncation $\mathcal{M}_{N,R}$ impose $f = Z = 0$ on its sole boundary S_R . In the first homotopy replace K everywhere by $K_a = aK$, $0 \leq a \leq 1$, retaining the definitions of $\mathcal{T}, F, V$ with that replacement, and put

$$F = \eta f, \quad \operatorname{div} V = u\Xi_a, \quad \Xi_a = \delta_0(\mathcal{T} + \eta a_w |df|) + \rho - m_0 C_N(\rho_0 + v\rho_1). \quad (4.4)$$

The second homotopy keeps $K = 0$ and replaces Ξ_0 by $s\Xi_0$, $0 \leq s \leq 1$. The weights and C_N do not vary in either homotopy. The endpoint $(a, s) = (0, 1)$ is common to them. Until the arbitrary-trial problem is introduced, the estimates concern smooth fixed points on a truncation with $t \geq -\epsilon$. The floor will be excluded on the boundary of the degree domain. Solvability for arbitrary trials will be established separately, without that restriction.

4.1 Deduction from the estimates

Fix N and a sufficiently large truncation radius R . Lemma 4.7 supplies a unique trace solution for each trial $z \in C^{1,\gamma}$ with zero outer value, without a floor restriction. It therefore defines the compact scalar homotopies in Equation (4.29). Their fixed points are smooth, as proved in

Section 4.8. On the open set in Equation (4.30), Propositions 4.3 and 4.6 bound the fixed-point norms. Choose its norm bound M larger than this bound. Lemma 4.2 excludes a floor minimum on the outer sphere and supplies the height hypothesis of Lemma 3.3, which excludes an interior minimum. Thus neither boundary of the degree domain contains a fixed point.

At $a = 0, s = 0$ the compact map is zero and its degree is one. The homotopy and excision argument in Section 4.8 therefore gives a fixed point at $a = 1, s = 1$ on every sufficiently large truncation. Proposition 4.6 makes these solutions locally compact as $R \rightarrow \infty$ at fixed N . The limit solves the filled system; the height and polynomial estimates pass to it, and Lemma 3.3 makes its floor inequality strict. Section 4.9 proves the remaining end estimates directly from the two equations. These steps give every assertion of Theorem 4.1. The local regularity statements below have their complete proofs in Appendix B; their successive application to f and Z is proved here.

4.2 Height comparison and the outer floor

The trace equation controls the normalized height before any derivative estimate. Its end comparison also bounds the outer normal derivative; this will prevent the floor from being reached on the truncating sphere.

Lemma 4.2 (Height and outer boundary estimates). *There are constants C and r_1 , independent of N, R and the homotopy parameters, such that every smooth solution of the trace equation in Equation (4.4) satisfies*

$$|\eta f| \leq C, \quad |\eta f| \leq C(r^{-b_1} - R^{-b_1}) \quad (r_1 \leq r \leq R).$$

For each fixed N there is $R_{\min}(N)$ such that $R \geq R_{\min}(N)$ and $Z = 0$ on S_R imply $t > -\epsilon$ there. The same conclusions hold in the second homotopy, where in fact $f = 0$.

Proof. At a positive maximum of f , $df = 0$, $A_\chi = I$, and

$$\eta f = \text{tr } K_a + l\Delta_g f \leq \text{tr } K_a.$$

The negative minimum gives the reverse bound. The tensor fields in the filling have a common pointwise bound, including along the product necks, so this proves the first assertion uniformly. If $K = 0$, both comparisons give $f = 0$.

On the end $K_a = 0$. At a point of contact with $H(r) = C(r^{-b_1} - R^{-b_1})$, considered as a test for $h = \eta f$, the two gradients agree. Thus the axial direction of A_χ is the radial direction. In the Euclidean metric the Hessian contraction is

$$A_\chi : \text{Hess } H = C b_1 r^{-b_1-2} ((b_1 + 1)\chi - k).$$

The error from $g - \delta = O_2(r^{2-n})$ has absolute value at most $C' r^{2-n} C b_1 r^{-b_1-2}$, uniformly for $0 < \chi \leq 1$. Since $b_1 + 1 < k$, the contraction is strictly negative for $r \geq r_1$, with r_1 independent of N . The height equation is

$$A_\chi : \text{Hess } h = \frac{\eta \sqrt{D}}{l} h.$$

Consequently H and $-H$ are respectively an upper and a lower comparison function. For $R \geq 2r_1$, their constant C can be chosen to dominate the already established height bound on S_{r_1} . The maximum principle then proves the displayed end estimate.

Taking the one-sided derivative at the outer zero boundary gives

$$|df|_{S_R} \leq C\eta^{-1}R^{-b_1-1}. \quad (4.5)$$

Tangential derivatives vanish there. For all real t we have $l(t) \leq e$. The identity $Z = t + (2k)^{-1} \log(1 + l(t)^2 |df|^2)$ therefore gives, on S_R ,

$$0 \geq t \geq -\frac{1}{2k} \log(1 + e^2 C^2 \eta^{-2} R^{-2b_1-2}).$$

Choosing $R_{\min}(N)$ sufficiently large proves the assertion. $\square$

Together, Lemmas 4.2 and 3.3 exclude every fixed point with $\min t = -\epsilon$. The height lemma supplies the bounds assumed by the scalar floor lemma and excludes the outer sphere by Equation (4.5). The scalar floor lemma then excludes an interior minimum in either homotopy. These exclusions concern the boundary of the degree domain; they make no assertion about solutions whose minima lie strictly below $-\epsilon$.

4.3 A polynomial upper bound before ellipticity

Above the floor, the implicit relation expresses the metric distortion directly in terms of Z . We therefore need an upper bound for this single scalar quantity. The proof first bounds a weighted integral over the region where Z is large, then tests the divergence equation with exponentials of Z and applies a Sobolev inequality on the graph of f . The bound for Z must be polynomial in N : the enclosure argument in Section 5 needs metric distortion at most $\exp(P(N))$ while N increases.

Proposition 4.3 (Uniform bounds on truncations). *Every smooth fixed point on $\mathcal{M}_{N,R}$ with $t \geq -\epsilon$ satisfies Equation (4.1), with the same polynomial bounds for all $R \geq R_{\min}(N)$ and all homotopy parameters.*

Proof. Write $\sigma = |df|$, $B = K_a(w, \cdot)$, $A = A_\chi$, and

$$d\mu_D = \sqrt{D} dV_g, \quad W = e^{(k-1)t} l, \quad u = W\sqrt{D} = l e^{kZ-t}.$$

The floor already implies

$$\ell \leq l \leq e, \quad t \leq Z, \quad D = e^{2k(Z-t)}, \quad \sigma \leq \ell^{-1} e^{k(Z+\epsilon)}. \quad (4.6)$$

Only the upper bound for Z remains to be proved.

High levels. On the support of m_0 , the floor implies $-\epsilon \leq t \leq -\epsilon + N^{-1}$ and $l \asymp \ell$. In particular,

$$\eta a_w \sigma = \frac{\eta}{l} (\sqrt{D} - D^{-1/2}).$$

The weights are bounded above uniformly, and C_N is polynomial. It follows from $D = e^{2k(Z-t)}$ that there is a positive polynomial threshold $Z_0 = P(N)$ such that

$$\Xi_a \geq \delta_0 \mathcal{T} + \rho + \frac{\delta_0}{2} \eta a_w \sigma \quad \text{when } Z > Z_0. \quad (4.7)$$

For example, it suffices that $e^{kZ_0} \geq C(1+C_N)\eta^{-1}$, after increasing the fixed constant; its logarithm has polynomial size. Away from the support of m_0 the inequality is immediate.

Choose a smooth nondecreasing ζ that vanishes on $(-\infty, Z_0]$ and equals one on $[Z_0 + 1, \infty)$, with $0 \leq \zeta' \leq C$. Testing the divergence equation with $\zeta(Z)$ gives

$$\int \zeta(Z) u \Xi_a = - \int \zeta'(Z) u \{k|dZ|_A^2 + \langle B, dZ \rangle_A\} \leq C \int_{\{Z_0 < Z < Z_0+1\} \cap \text{supp } K} u |K|^2.$$

The integration has no boundary term since $Z = 0 < Z_0$ on S_R . On the last strip $u = l e^{kZ-t} \leq \exp(P(N))$. The support of K consists of a fixed exterior compact set and the enlarged filling; it has volume $O(1 + N)$. Thus

$$\int_{\{Z > Z_0+1\}} u (\delta_0 \mathcal{T} + \rho + \frac{1}{2} \delta_0 \eta a_w \sigma) dV_g \leq \exp(P(N)). \quad (4.8)$$

This estimate uses no upper bound for σ . It controls the high levels of Z with exactly the volume and slope weights needed on the graph. We next convert it into the initial integrability for an iteration on fixed base patches.

Fix $0 < \alpha \leq k - 1$, independent of N . Cover the enlarged core and a fixed extra end annulus by base coordinate patches of uniformly controlled size and geometry. The number of patches is $O(1 + N)$, and ρ has a uniform positive lower bound on them. We claim that on each such patch Q ,

$$\int_Q e^{\alpha Z} d\mu_D \leq \exp(P(N)). \quad (4.9)$$

Below $Z_0 + 1$, Equation (4.6) proves this directly. Above that level, when $D \leq 2$ use

$$\frac{e^{\alpha Z} \sqrt{D}}{u} = l^{-1} e^{(\alpha-k+1)t} D^{\alpha/(2k)} \leq \exp(P(N)).$$

When $D \geq 2$, use instead

$$u \eta a_w \sigma = \eta e^{(k-1)t} (D - 1), \quad e^{\alpha Z} \sqrt{D} = e^{\alpha t} D^{1/2 + \alpha/(2k)}.$$

The ratio of the second expression to the first is at most $C \eta^{-1} e^{(\alpha-k+1)t} \leq \exp(P(N))$ because $1/2 + \alpha/(2k) \leq 1$ and $t \geq -\epsilon$. Equation (4.8) proves the claim.

Sobolev inequality on the product graph. We use the ordinary graph of f in $(Q \times \mathbb{R}, g + dz^2)$ so that its ambient geometry depends only on the fixed background patch. The warp $l(t)$ has no a priori derivative bound yet. We compare the ordinary graph's measure and inverse metric with $d\mu_D$ and A , and use $\mathcal{T}$ to control its mean curvature. The ordinary graph has measure and inverse metric

$$\sqrt{1 + \sigma^2} dV_g, \quad A_* = I - \frac{df \otimes df}{1 + \sigma^2}.$$

The comparison uses the same gradient axis for both inverse metrics. If $d_* = (1 + \sigma^2)^{-1}$ is the axial eigenvalue of A_* , then

$$\ell^2 d \leq d_* \leq e^2 d, \quad \frac{k}{k+N} d \leq \chi \leq d, \quad e^{-1} \leq \frac{\sqrt{1 + \sigma^2}}{\sqrt{D}} \leq \ell^{-1}.$$

The transverse eigenvalues are all one. These relations compare both the quadratic gradient norms and the volume measures with factors $\exp(P(N))$, uniformly in the possibly unbounded slope σ . The comparison is pointwise and uses no derivative of l .

Its mean curvature is $(1 + \sigma^2)^{-1/2} \text{tr}_{A_*} \text{Hess } f$. Write $K_a + H^f$ in its transverse and axial blocks. The transverse contraction is controlled by $|T|$, and the axial coefficient of A_* is at most $e^2 d$. Also $\sqrt{D}/(l\sqrt{1 + \sigma^2}) \leq \exp(P(N))$. Proposition 3.2 therefore gives

$$|H_{\text{graph}}| \leq \exp(P(N))(1 + \sqrt{\mathcal{T}}).$$

Extend the metric of a slightly larger fixed base patch to a closed manifold and use Nash's smooth isometric embedding theorem [36]; then take its product with the line. The additional second fundamental form is uniformly bounded. Its trace on any orthonormal n -frame tangent to the graph is bounded by n times that bound, so the ambient embedding adds a fixed constant to the graph's mean-curvature bound, independently of its slope. Such embeddings can be chosen from finitely many fixed models for the cap, transitions, and product necks. The Michael–Simon inequality [35], in the formulation of [43, Chapter 4, Section 6, Theorem 6.7], applied to compactly supported functions on this smooth n -dimensional graph, then gives, after its usual L^1 to L^2 conversion,

$$\left(\int |\varphi|^{2\kappa_n} d\mu_D \right)^{1/\kappa_n} \leq \exp(P(N)) \int (|d\varphi|_A^2 + (1 + \mathcal{T})\varphi^2) d\mu_D, \quad \kappa_n = \frac{n}{n-2}. \quad (4.10)$$

To see the conversion explicitly, apply the L^1 inequality to $|\varphi|^{2(n-1)/(n-2)}$ and use Cauchy–Schwarz on its derivative and mean-curvature terms. The resulting common $L^{2n/(n-2)}$ factor cancels. No bound for the height or area of the graph is required.

Exponential testing. The graph inequality still contains $\mathcal{T}$ on its right side. The divergence equation controls that term together with the gradient of an exponential of Z ; this is the estimate that closes the iteration. Globally $\Xi_a \geq \delta_0 \mathcal{T} - P(N)$. Let ψ be a base cutoff compactly supported in Q and test the equation with $\psi^2 e^{2qZ}/W$. Put $\beta_W = d \log W = (k-1+p)dt$. Since $A \leq A_d$, scalar coercivity gives $|\beta_W|_A^2 \leq P(N)\mathcal{T}$. Integration by parts yields exactly

$$\int \psi^2 e^{2qZ} \Xi_a d\mu_D = - \int e^{2qZ} (2q\psi^2 dZ - \psi^2 \beta_W + 2\psi d\psi) \cdot A(kdZ + B) d\mu_D.$$

Young's inequality bounds the $\beta_W \cdot A dZ$ term by $\delta_0 \mathcal{T}/8 + P(N)|dZ|_A^2$, the $\beta_W \cdot AB$ term by $\delta_0 \mathcal{T}/8 + P(N)$, and the $qB \cdot A dZ$ term by a fixed fraction of $kq|dZ|_A^2 + Cq$. The cutoff terms cost a further fixed fraction of that gradient term and $P(N)(\psi^2 + |d\psi|^2)$. Consequently there is a polynomial $q_*(N) \geq \max\{1, \alpha\}$ such that, for $q \geq q_*(N)$,

$$\int e^{2qZ} \psi^2 (\mathcal{T} + q|dZ|_A^2) d\mu_D \leq P(N)(1+q) \int e^{2qZ} (\psi^2 + |d\psi|_g^2) d\mu_D. \quad (4.11)$$

Only dt occurs in this test; it does not differentiate p .

Apply Equation (4.10) with $\varphi = \psi e^{qZ}$ and then use Equation (4.11). For nested base patches $Q' \Subset Q$ separated by a coordinate distance $b \leq 1$, this gives

$$\|e^Z\|_{2\kappa_n q, Q'} \leq [e^{P(N)}(1+q)^2 b^{-2}]^{1/(2q)} \|e^Z\|_{2q, Q}.$$

All norms here use $d\mu_D$. Iterate with $q_i = \kappa_n^i q_*$ and geometrically decreasing patch gaps. The sums of q_i^{-1} and $i q_i^{-1}$ converge, so

$$\sup_{Q'} e^Z \leq e^{P(N)} b^{-C} \|e^Z\|_{2q_*, Q}, \quad (4.12)$$

where C is independent of N .

Finally interpolate the norm on the right with Equation (4.9):

$$\|e^Z\|_{2q_*, Q} \leq \left(\sup_Q e^Z\right)^\theta e^{P(N)}, \quad \theta = 1 - \frac{\alpha}{2q_*} < 1.$$

Choose expanding patches Q_j , all inside one fixed larger patch, with gaps comparable to 2^{-j} . The logarithms $M_j = \log \sup_{Q_j} e^Z$ satisfy $M_j \leq \theta M_{j+1} + P(N) + Cj$. Iteration gives

$$M_0 \leq \theta^m M_m + \sum_{j=0}^{m-1} \theta^j (P(N) + Cj) \leq \theta^m M_m + P(N).$$

The last polynomial is justified by $(1 - \theta)^{-1} = 2q_*/\alpha \leq P(N)$. For each individual smooth solution the suprema on the largest patch are finite, so the first term tends to zero. We have proved $Z \leq P(N)$ on the enlarged core and the drift support. A larger maximum outside that set would have $K_a = 0$, $dZ = 0$, and, by Equation (4.7), $\Xi_a > 0$; at the same point $u^{-1} \operatorname{div}(kuA\nabla Z) = kA : \operatorname{Hess} Z \leq 0$, a contradiction.

For the second homotopy $f = 0$. A positive maximum above the penalty band has right side $s\rho > 0$ if $s > 0$, which is impossible. When $s = 0$ the solution is zero. This proves its upper bound as well. Equation (4.6) and the height bound prove the remaining bounds in Equation (4.1). In particular,

$$d \geq e^{-2k(P(N)+\epsilon)}, \quad \chi \geq \frac{k}{k+N} e^{-2k(P(N)+\epsilon)}.$$

The same bounds give positive upper and lower bounds for u at each fixed N . Thus both A_χ and kuA_χ are uniformly elliptic, with constants independent of the truncating radius and the homotopy parameter. These are the ellipticity bounds used in the following scalar estimates. $\square$

4.4 A scalar gradient estimate with a measurable axial coefficient

The upper bound has made both scalar equations uniformly elliptic at each fixed N . Regularity still has to pass between them in a specific order. Dividing the trace equation by $l/\sqrt{D}$ gives

$$A_\chi : \operatorname{Hess}_g f = \frac{\sqrt{D}}{l} (\eta f - \operatorname{tr}_{A_\chi} K_a).$$

Its right side and Df are bounded, but its coefficients depend on Z , for which we have no continuity modulus. Schauder theory does not yet apply.

The gradient direction accounts for the matrix structure; the remaining unknown eigenvalue χ can be treated as merely measurable. The next lemma gives both a Hölder bound for Df and a local mass bound for $|D^2 f|^2$. This second conclusion is needed because the divergence source for Z contains $|D^2 f|^2$: the mass bound will give Hölder continuity of Z through the following scalar lemma. Once both Z and Df have a modulus, scalar Schauder estimates can begin. All constants in this regularity argument may depend arbitrarily on fixed N ; the bounds needed as N increases were established above.

We prove the gradient estimate in two regimes. Absorption using $|I - A| < 1$ first gives $W^{2,s}$ control for some $s > 2$. A flux formed by applying the trace-reversed Hessian to the gradient then shows that either the gradient norm drops on a smaller ball or the solution is close to an affine function with nonzero gradient. In the latter regime we freeze the direction while retaining the measurable axial coefficient; the axial derivative satisfies a scalar divergence equation. Iterating these two alternatives gives gradient continuity and the required Hessian mass bound.

Lemma 4.4 (Gradient estimate with a measurable axial coefficient). *Let $n \geq 3$ and $0 < c \leq 1$. Let Q be a smooth metric coordinate ball, or a smooth half-ball with constant Dirichlet data on its face. Suppose that u is C^2 up to the face and satisfies almost everywhere*

$$A : \text{Hess}_g u = G, \quad A = \text{id} - (1 - \chi)e \otimes e, \quad c \leq \chi \leq 1, \quad e = \frac{\nabla_g u}{|\nabla_g u|_g} \quad \text{where } \nabla_g u \neq 0. \quad (4.13)$$

At a zero of the gradient, e may be any measurable unit vector. Assume $|\nabla_g u|_g + |G| \leq M$. On every smaller ball, including its portion of the Dirichlet face, there are estimates

$$[Du]_{C^\alpha} \leq C, \quad \int_{B_r(x) \cap Q} |\text{Hess}_g u|_g^2 dV_g \leq Cr^{n-2+2\alpha}. \quad (4.14)$$

Here $\alpha \in (0, 1)$ depends only on n, c , and C depends on n, c, M , the fixed coordinate geometry and the distance from the other chart boundaries. No modulus of continuity of χ enters these constants.

The complete proof is in Appendix B.1.

4.5 A bounded scalar equation with quadratic gradient growth

The trace estimate now supplies the local mass control needed for the second equation. Its source has quadratic growth in DZ and also contains $|D^2 f|^2$. The latter need not yet have a pointwise bound uniform over solutions: the estimate $\int_{B_r} |D^2 f|^2 \leq Cr^{n-2+2\alpha}$ from the preceding lemma is the information available at this stage.

The following bounded-solution estimate uses exactly such a mass bound. Two exponential changes of variable absorb the quadratic DZ term, and the mass bound makes the remaining source potential small on short scales. Oscillation decay then gives a Hölder modulus for Z while its principal coefficients are still treated as measurable.

Lemma 4.5 (Hölder estimate for quadratic gradient growth). *Let $n \geq 3$ and let Q be a Euclidean coordinate ball, or a smooth flattened half-ball with constant Dirichlet value on its face. Suppose a bounded classical function Z satisfies weakly*

$$\begin{aligned} -\partial_i(A^{ij}\partial_j Z) &= \partial_i B^i + S, & \lambda I &\leq A \leq \Lambda I, & A &= A^\top, \\ |B| &\leq C_0, & |S| &\leq C_0(1 + |DZ|^2 + F_0). \end{aligned} \quad (4.15)$$

where A is measurable and $F_0 \geq 0$ is bounded for each individual solution. Assume, for some $a \in (0, 1)$ and $r_0 > 0$, and every ball of radius $0 < r \leq r_0$ with center in the coordinate patch, that the measure $\nu = (1 + F_0) dx$ obeys

$$\nu(B_r \cap Q) \leq C_1 r^{n-2+a}. \quad (4.16)$$

Then Z has a uniform C^b bound on smaller patches, including the constant-Dirichlet face, for some $b \in (0, a)$. Its exponent and bound depend only on $n, \lambda, \Lambda, C_0, C_1, a, r_0$, the bound for $|Z|$, the fixed chart geometry and the distance from the remaining patch boundary. They use neither a modulus of continuity for A nor a uniform pointwise bound for F_0 or $D^2 Z$.

The complete proof is in Appendix B.2.

4.6 Regularity of the coupled fixed points

We apply the two local lemmas in a definite order. The trace equation first controls Df and the local mass of $|D^2 f|^2$. The scalar divergence lemma then controls the oscillation of Z . These two Hölder bounds permit scalar Schauder estimates, followed by an absorption of the remaining quadratic gradient term. Each step has constants uniform in the outer truncation at fixed N .

Proposition 4.6 (Local regularity). *Fix N with the restrictions in Equation (3.1). Smooth solutions of either homotopy on $\mathcal{M}_{N,R}$ with $t \geq -\epsilon$ have bounds of every fixed order on coordinate balls and on outer Dirichlet half-balls of a sufficiently small fixed size. Their size and the constants may depend arbitrarily on N , but are independent of $R \geq R_{\min}(N)$ and of the homotopy parameter. In particular their $C^{1,\gamma}$ norms are bounded on each fixed truncation, for any chosen $0 < \gamma < 1$.*

Proof. 1. The trace estimate and divergence form. Fix N and a member of the homotopy. Proposition 4.3 gives uniform bounds for $|f|$, $|df|$, and $|Z|$, and uniform ellipticity on coordinate balls of a fixed small radius. The radius and constants may depend on N , but can be chosen independently of the outer truncation. Lemma 4.4, applied to the trace equation, gives some $a \in (0, 1)$ such that

$$[df]_{C^a} \leq C, \quad \int_{B_r \cap \mathcal{M}_{N,R}} |D^2 f|^2 dx \leq Cr^{n-2+2a}, \quad 0 < r \leq r_0. \quad (4.17)$$

Here and below the boundary version uses a smooth half-ball with constant Dirichlet data; coordinate and covariant Hessians differ by a bounded quantity. Each occurrence of a ball in a mass estimate allows all centers in the coordinate patch, not just one fixed center.

The implicit relation defining t has derivative $1 + pv/k \geq 1$ with respect to t . On the bounded range now under consideration it therefore defines a smooth function of (x, Z, Df) , whose relevant partial derivatives are bounded. In particular,

$$|Dt| \leq C(1 + |DZ| + |D^2 f|). \quad (4.18)$$

All the expressions in the equations are smooth before the block decomposition, including at $df = 0$. Since $\mathcal{T}$ is quadratic in H^f , dt , and K , Equation (4.18) and the bounded zeroth- and first-order quantities imply the following coordinate form of the second equation:

$$-\partial_i(A^{ij}\partial_j Z) = \partial_i B^i + S, \quad \lambda I \leq A \leq \Lambda I, \quad |B| \leq C, \quad |S| \leq C(1 + |DZ|^2 + |D^2 f|^2). \quad (4.19)$$

The matrix A is symmetric. With volume densities included, A represents kuA_χ , B represents $uA_\chi K(w, \cdot)$, and $S = -\sqrt{\det g} u \Xi$. The same bounds hold uniformly along both stages of the homotopy; scaling Ξ towards zero improves its absolute-value bound. No continuity modulus for A is being used yet.

Apply Lemma 4.5 with $F_0 = |D^2 f|^2$. Its coefficient and source hypotheses are precisely Equation (4.19). On patches of size at most one, Equation (4.17) gives $\nu(B_r) \leq Cr^{n-2+a}$ for $\nu = (1 + |D^2 f|^2) dx$. The lemma therefore gives a uniform C^b bound for Z for some $0 < b < a$, up to the constant outer Dirichlet face as well. Both Z and Df now have the continuity needed to improve the trace equation by Schauder theory.

2. Absorbing the quadratic gradient term. The coefficients of the trace equation are smooth functions of (x, Z, Df) . The preceding estimate and Equation (4.17) therefore give a uniform

C^c modulus for those coefficients and the trace right side, with $c = \min(a, b) > 0$. Interior and Dirichlet Schauder estimates give

$$\|f\|_{C^{2,c}} \leq C \quad (4.20)$$

on smaller patches of the same uniform type. This use of Schauder requires only the now established Hölder modulus of (Z, Df) .

Expand the divergence equation on each original ball or half-ball. Since $A = A(x, Z, Df)$ and $B = B(x, Z, Df)$, their derivatives consist of bounded terms, terms linear in DZ , and terms linear in D^2f . Equation (4.20) and the formula for Dt show that the expanded equation has the form

$$A^{ij}D_{ij}Z = H, \quad |H| \leq C(1 + |DZ|^2), \quad \|A\|_{C^c} \leq C. \quad (4.21)$$

The bound on the Hölder norm of A does not presuppose a bound for DZ . Coordinate lower-order terms are included in H .

Fix any finite $s > n$. Freezing the now Hölder principal matrix on sufficiently small balls gives the interior and flat-Dirichlet $W^{2,s}$ estimates with uniformly bounded coefficient in front of $\|H\|_s$. On nested patches $Q_r \subset Q_{2r}$ these give

$$\|D^2Z\|_{s,Q_r} \leq C_s \|DZ\|_{2s,Q_{2r}}^2 + C_{s,r}. \quad (4.22)$$

The last constant includes the bounded height, the constant term in Equation (4.21), and the scaled cutoff costs. For a ball or a fixed-shape half-ball the scaled interpolation inequality is

$$\|DZ\|_{2s,Q}^2 \leq C_s \operatorname{osc}_Q Z \|D^2Z\|_{s,Q} + C_s r^{n/s-2} (\operatorname{osc}_Q Z)^2, \quad (4.23)$$

where Q has size r . Indeed apply the usual second-order interpolation inequality on the unit ball or half-ball after subtracting a constant and rescaling. On a half-ball, the constant Dirichlet value belongs to the closed range of Z , so it can be used for this subtraction. The constants depend only on the uniform chart shapes, s , and the dimension.

For clarity, the absorption is local and does not lose the number of patches in a large truncation. For a fixed sufficiently small r , let

$$\mathcal{M}_r = \sup_x \|D^2Z\|_{s,Q_r(x)}.$$

It is finite for each individual classical solution on a truncation. The uniform local geometry allows each $Q_{2r}(x)$ to be covered by at most J_0 patches of size r , with J_0 independent of the outer radius. Thus Equations (4.22) and (4.23) imply

$$\mathcal{M}_r \leq C_s J_0^{1/s} \sup_x \operatorname{osc}_{Q_{2r}(x)} Z \mathcal{M}_r + C_{s,r}.$$

Choose r small enough, using the uniform C^b bound already proved, that the displayed coefficient of $\mathcal{M}_r$ is at most $1/2$. This gives a uniform local $W^{2,s}$ bound up to the outer face, independently of the total number of patches. Sobolev embedding now bounds Z in $C^{1,d}$ for $d < 1 - n/s$.

3. Bootstrap. The trace equation consequently bounds f in $C^{3,d'}$ for some $d' > 0$. Its coefficients now have one Hölder derivative; the expanded second equation has Hölder right side, giving $Z \in C^{2,d'}$. Repetition yields bounds of every fixed order, since the background data are smooth. More explicitly, once $Z \in C^{j,d'}$ and $f \in C^{j+1,d'}$ for $j \geq 1$, the trace equation first gives $f \in C^{j+2,d'}$, and the divergence equation then gives $Z \in C^{j+1,d'}$. Shrinking the exponent once at the first step suffices. The argument applies initially to the $C^{2,\gamma}$ fixed points constructed below: their individual Hessian norms are already finite, and smoothness is the conclusion of the bootstrap. All constants in this paragraph may depend arbitrarily on fixed N and the chosen derivative order.

□

4.7 The trace equation for an arbitrary trial

The degree argument requires a map defined beyond the region $t > -\epsilon$. The next lemma supplies it. Its estimates are separate from Proposition 4.3: here a trial Z is prescribed, and no lower bound for its associated t is assumed.

Lemma 4.7 (The trace equation for prescribed conformal data). *Let $\mathcal{D}$ be a compact smooth n -dimensional Riemannian manifold with nonempty smooth boundary, with $n \geq 3$ and $k = n - 1$, and let K be a smooth symmetric tensor on $\mathcal{D}$. Fix $N > k$, $0 < \gamma < 1$, and a positive constant η . Use $p(t) = N\vartheta(Nt)$ and $l(t) = \exp(\int_0^t p(s) ds)$, with the smooth cutoff ϑ of Equation (3.1), and define D , t , A_χ by Equations (3.2) and (3.3). Put $K_a = aK$. For every $z \in C^{1,\gamma}(\mathcal{D})$ and $a \in [0, 1]$, the Dirichlet problem*

$$\mathrm{tr}_{A_\chi} \left(K_a + \frac{l}{\sqrt{D}} \mathrm{Hess} f \right) = \eta f, \quad f|_{\partial\mathcal{D}} = 0, \quad t + \frac{1}{2k} \log(1 + l(t)^2 |df|^2) = z$$

has a unique solution $f = f_a[z] \in C^{3,\gamma}$. The solution map is continuous from $[0, 1] \times C^{1,\gamma}(\mathcal{D})$ to $C^{3,\gamma}(\mathcal{D})$ and maps bounded sets to bounded sets. All constants may depend on $N, \eta, \mathcal{D}, g, K$ and the prescribed bound for z . There is no restriction on the minimum of the implicitly determined t .

Proof. 1. Coefficient growth without a floor. Let $b = \sqrt{D}/l$. On a bounded trial range the coefficients satisfy, for every $\sigma = |df|$,

$$b \leq C(1 + \sigma), \quad \frac{c}{1 + \sigma} \leq \chi \leq 1. \quad (4.24)$$

We verify the bounds also at large slope, where t has no fixed floor. The implicit relation implies $t < 0$ when σ is sufficiently large. There $p = N$, $l = e^{Nt}$, and

$$e^{2kz} = e^{2kt} + \sigma^2 e^{2(k+N)t}.$$

Uniformly on a bounded z range, as $\sigma \rightarrow \infty$,

$$t = \frac{kz - \log \sigma}{k + N} + o(1), \quad d \asymp \sigma^{-2k/(k+N)}, \quad \chi \asymp \frac{k}{k + N} \sigma^{-2k/(k+N)}.$$

The exponent is less than one because $N > k$. Moreover $b = \sigma/\sqrt{v} \asymp \sigma$ there. The remaining bounded slope range is compact in (z, σ) , and the coefficients are smooth and positive on it. This proves Equation (4.24).

2. A global gradient bound. Continue from $\Delta_g f = f$ by solving, for $0 \leq q \leq 1$,

$$A_q : \mathrm{Hess} f - c_q f = -qb \mathrm{tr}_{A_\chi} K_a, \quad A_q = (1 - q)I + qA_\chi, \quad c_q = 1 - q + q\eta b. \quad (4.25)$$

The scalar c_q is positive. Extrema give a common height bound $\|f\|_\infty \leq H$, since

$$\frac{qb |\mathrm{tr}_{A_\chi} K_a|}{c_q} \leq \frac{C}{\eta}.$$

On this height range the right side of the equation written as $A_q : \mathrm{Hess} f = G(x, f, df)$ satisfies, for a covector ξ ,

$$|G(x, f, \xi)| \leq C(1 + |\xi|), \quad (4.26)$$

uniformly in q, a . The eigenvalues of A_q are one in the transverse directions and $\chi_q = 1 - q + q\chi \geq c/(1 + |\xi|)$ along ξ .

Here is a gradient estimate using precisely this lower axial bound. Choose K_0 sufficiently large, then a positive distance d_0 smaller than the boundary tubular radius, the injectivity radius of a smooth extension of the compact domain, and $(2K_0)^{-1}$. For $M > 0$ set

$$\psi(d) = K_0^{-1} \log(1 + K_0 M d), \quad \psi'' = -K_0(\psi')^2. \quad (4.27)$$

By increasing M we ensure both $\psi' \geq 1$ on $[0, d_0]$ and $\psi(d_0) > 2H$. Indeed $\psi'(d_0)$ tends to $(K_0 d_0)^{-1} > 2$, while $\psi(d_0)$ tends to infinity.

First compare with $\pm\psi(d(x, \partial\mathcal{D}))$ in the boundary collar. At a contact, the gradient is normal and has length $\sigma = \psi'$. The tangential Hessian contraction costs at most $C\sigma$, whereas

$$\chi_q \psi'' \leq -\frac{cK_0 \sigma^2}{1 + \sigma} \leq -\frac{cK_0}{2} \sigma.$$

Choose K_0 large enough to dominate the collar curvature and Equation (4.26). The positive barrier is a strict supersolution; the negative one is a strict subsolution. On the inner edge of this collar they dominate $\pm f$ by the height bound, and both vanish at the boundary. Therefore

$$|f(x)| \leq \psi(d(x, \partial\mathcal{D})) \quad \text{when } d(x, \partial\mathcal{D}) \leq d_0. \quad (4.28)$$

Consider now $f(x) - f(y) - \psi(d(x, y))$ for pairs with $0 \leq d(x, y) \leq d_0$, using the distance of the smooth ambient extension. It is nonpositive on the diagonal, for $d = d_0$, and when either endpoint lies on the boundary, by Equation (4.28) and the monotonicity of ψ . Suppose it had a positive interior maximum. The endpoint gradients are parallel along their unique short joining geodesic and have the same length $\sigma = \psi' \geq 1$. Write e for their common parallel direction. Varying one endpoint in the axial direction gives

$$\text{Hess } f_x(e, e) \leq \psi'', \quad \text{Hess } f_y(e, e) \geq -\psi''.$$

Varying both endpoints in parallel transverse directions and summing the second-variation inequalities gives

$$\text{tr}_{e^\perp} \text{Hess } f_x - \text{tr}_{e^\perp} \text{Hess } f_y \leq C d(x, y) \sigma.$$

For completeness, the simultaneous endpoint variation is generated by a parallel vector field along the geodesic; its index form is bounded above by $C d(x, y)$ since the ambient curvature is bounded. This is the displayed estimate after multiplication by ψ' . The transverse eigenvalues are exactly one at both endpoints, so

$$\begin{aligned} A_{q,x} : \text{Hess } f_x - A_{q,y} : \text{Hess } f_y &\leq C d(x, y) \sigma + (\chi_{q,x} + \chi_{q,y}) \psi'' \\ &\leq C d_0 \sigma - c K_0 \sigma. \end{aligned}$$

The right sides of Equation (4.25) differ by a quantity at least $-2C(1 + \sigma)$. A sufficiently large K_0 contradicts this. Notice that no estimate for $\chi_{q,x} - \chi_{q,y}$ was needed: both signed axial Hessian bounds have the favorable sign. We conclude $|f(x) - f(y)| \leq \psi(d(x, y))$ and hence $|df| \leq \psi'(0) = M$.

3. Continuation and regularity. Equation (4.24) now gives uniform ellipticity throughout the continuation. Its normalized forcing is bounded. Lemma 4.4 gives a common positive Hölder exponent for df , including the boundary. Since the coefficients are smooth functions of (x, z, df) ,

the Dirichlet Schauder estimate first gives $f \in C^{2,\gamma_0}$ for a possibly smaller exponent. The resulting Lipschitz bound for df then makes those coefficients $C^{0,\gamma}$, so the estimate improves to $C^{2,\gamma}$. Differentiating the equation once gives $C^{3,\gamma}$ estimates: the trial is $C^{1,\gamma}$ and all differentiated coefficient and source terms are controlled. These scalar estimates are uniform along the continuation; they are the interior and smooth Dirichlet estimates of [22] on a finite coordinate covering.

The linearization in f is a uniformly elliptic scalar operator with bounded Hölder first-order coefficients and zero-order coefficient $-c_q < 0$. Its homogeneous Dirichlet kernel is zero by the maximum principle. The scalar linear Dirichlet continuity theorem, starting from $\Delta_g - 1$ and using the Schauder estimate, makes it invertible from $C_0^{2,\gamma}$ to $C^{0,\gamma}$. Here $C_0^{j,\gamma}(\mathcal{D})$ denotes the subspace of $C^{j,\gamma}(\mathcal{D}; \mathbb{R})$ with zero boundary trace, not a condition on normal derivatives. The implicit function theorem gives openness of the nonlinear continuation set; the bounds and compactness give closedness. The initial solution is $f = 0$, hence existence follows at $q = 1$.

4. *Uniqueness and continuity.* For uniqueness compare two solutions at a positive maximum of their difference. Their gradients agree there, so their coefficient matrices and their positive factors b agree; Hessian ordering and the strictly negative zero-order term give a contradiction. For continuity in the stated norm, regard the normalized trace operator as a map

$$[0, 1] \times C^{1,\gamma} \times C_0^{3,\gamma} \longrightarrow C^{1,\gamma}.$$

It is continuously differentiable in the trial and solution variables, since its coefficient functions are smooth on every bounded jet range. At a solution its linearized principal, first-order, and zero-order coefficients are $C^{1,\gamma}$. The same maximum principle and the one-derivative-higher Dirichlet Schauder estimate therefore make that linearization an isomorphism from $C_0^{3,\gamma}$ onto $C^{1,\gamma}$. The implicit function theorem supplies local continuous dependence into $C^{3,\gamma}$; uniqueness joins these local maps. The a priori estimates already proved give the bounded-set assertion. $\square$

4.8 A compact map and the two homotopies

We now return to $\mathcal{D} = \mathcal{M}_{N,R}$ and $\eta = \ell^{3/2}$. Fix N, R with $R \geq R_{\min}(N)$, and let

$$X = \{z \in C^{1,\gamma}(\mathcal{M}_{N,R}) : z|_{S_R} = 0\}, \quad 0 < \gamma < 1.$$

For a trial $z \in X$, solve $f = f_a[z]$ by Lemma 4.7. Evaluate t, D, u, A_χ, w and $\mathcal{T}$ using (f, z) , including the derivatives of t obtained by differentiating its implicit definition. Define $T_a(z) = Z_{\text{out}}$ to be the solution of the *linear* Dirichlet problem

$$\operatorname{div}(kuA_\chi \nabla Z_{\text{out}}) = u\Xi_a - \operatorname{div}(uA_\chi K_a(w, \cdot)), \quad Z_{\text{out}}|_{S_R} = 0. \quad (4.29)$$

Only the displayed gradient of Z_{out} is unfrozen. The source Ξ_a is evaluated at the input z , not at the output.

On bounded subsets of X , the trace lemma gives bounded $C^{3,\gamma}$ norms for f . Consequently the principal coefficient kuA_χ is $C^{1,\gamma}$ and uniformly positive definite, the drift flux is $C^{1,\gamma}$, and $u\Xi_a$ is $C^{0,\gamma}$. The linear scalar Dirichlet theorem and Schauder estimate therefore give a unique $C^{2,\gamma}$ output. The map $T_a : X \rightarrow X$ is continuous and compact, uniformly on bounded sets and $a \in [0, 1]$. The same construction, with $a = 0$ and source $su\Xi_0$, defines the second compact homotopy $T_{0,s}$. Its trace solution is zero for every trial, since $K = 0$.

Every fixed point bootstraps to smoothness. Indeed the first Schauder step just described gives $z \in C^{2,\gamma}$; the trace equation then gives $f \in C^{4,\gamma}$, and repeated application to the second

equation and then the first gives arbitrary derivatives. Thus the a priori estimates proved above apply to all fixed points.

For the first homotopy let $\lambda = a$ and $f_\lambda[z] = f_a[z]$. For the second let $\lambda = s$ and $f_\lambda[z] = f_0[z] = 0$ for every trial $z \in X$. In either case, $t[z, df_\lambda[z]]$ denotes the pointwise solution of Equation (3.2) with $Z = z$ and graph differential $df_\lambda[z]$. Set

$$\mathcal{U}_\lambda = \{z \in X : \min_{\mathcal{M}_{N,R}} t[z, df_\lambda[z]] > -\epsilon, \|z\|_{C^{1,\gamma}} < M\}. \quad (4.30)$$

The associated set of parameter–trial pairs is

$$\mathcal{U} = \{(\lambda, z) \in [0, 1] \times X : z \in \mathcal{U}_\lambda\}.$$

It is relatively open in $[0, 1] \times X$ because the trace solution map and the implicit function defining t are continuous. Choose M larger than the common $C^{1,\gamma}$ bound for all fixed points with $t > -\epsilon$ supplied by Proposition 4.6. There is no fixed point on the norm boundary, and the floor lemmas exclude one on the other boundary.

Here is the precise reason degree may be used on these moving sets. The fixed points in the closure of $\mathcal{U}$ form a compact subset of $[0, 1] \times X$: the parameter interval is compact, the trial norms are bounded, and compactness of the homotopy gives a convergent subsequence of every sequence of such fixed points. Its limit cannot lie on either excluded boundary. Near any one parameter, take a fixed open neighborhood of that compact fiber whose closure remains inside $\mathcal{U}_\lambda$ for all sufficiently nearby parameters; shrink the parameter interval to exclude other fixed points outside this neighborhood, using compactness. Excision and ordinary fixed-domain homotopy invariance identify the degrees at those nearby parameters. A finite chain of such intervals identifies the degree at the two ends. These are the usual excision and homotopy properties of Leray–Schauder degree [17, Chapter 2].

At the endpoint $a = 0, s = 0$, Equation (4.29) has zero source and zero boundary value, so $T_{0,0}(z) = 0$ for every trial. Its sole fixed point is zero, with $t = 0 > -\epsilon$, and its degree is one. First follow the second homotopy to $s = 1$, then the first to $a = 1$. The degree remains one, yielding a smooth solution of the desired filled system on every sufficiently large truncation.

4.9 Exhaustion, end decay, and the energy flux

The degree argument gives solutions on every sufficiently large truncation. Uniform local estimates let the outer boundary escape while N remains fixed. The positive zeroth-order term in the trace equation then gives exponential decay of the graph; a scalar change of variable converts the end equation for Z to a Poisson estimate. These are the precise asymptotics needed to evaluate the energy.

Completion of the proof of Theorem 4.1. Keep N fixed and let $R \rightarrow \infty$. The estimates of Proposition 4.6 on compact sets are independent of R . A diagonal subsequence therefore converges smoothly on every compact subset of $\mathcal{M}_N$ to a solution (f, Z) . The polynomial bounds and height comparisons survive. They give $t \geq -\epsilon$; equality would be an interior global minimum and contradict Lemma 3.3.

We prove the remaining end estimates on the truncation solutions, uniformly in R at this fixed N ; the estimates then pass to the smooth limit just obtained.

On the end $K = 0$ and the trace equation reads

$$A_\chi : \text{Hess } f - \frac{\eta\sqrt{D}}{l}f = 0, \quad \frac{\eta\sqrt{D}}{l} \geq \frac{\eta}{e} > 0. \quad (4.31)$$

For small $a_N > 0$ and a sufficiently large fixed $r_2(N)$, $v_N(r) = e^{-a_N(r-r_2)}$ obeys

$$A_\chi : \text{Hess } v_N - \frac{\eta\sqrt{D}}{l}v_N < 0 \quad (r \geq r_2).$$

In fact $0 < A_\chi \leq I$ and the end metric bounds give $A_\chi : \text{Hess } v_N \leq C(a_N^2 + a_N/r)v_N$; choose $Ca_N^2 < \eta/(4e)$ and then r_2 large. Multiples of $\pm v_N$ dominate f at S_{r_2} and at the outer zero sphere. The maximum principle for the operator in Equation (4.31), with its coefficients evaluated at the solution, proves exponential decay uniformly on all truncations. For the derivative conclusion, treat Equation (4.31) as a homogeneous linear equation with its coefficients evaluated at this solution. The local regularity bounds already proved give uniform coefficient bounds of every required order at fixed N . On a fixed-size ball, or an outer boundary half-ball with zero Dirichlet data, the linear scalar estimates give

$$\|f\|_{C^j(Q')} \leq C_{N,j}\|f\|_{C^0(Q)} \quad (Q' \Subset Q).$$

The radius changes by a bounded amount inside such a patch, so the exponential C^0 bound on Q gives the same exponential rate on Q' . Thus every derivative of f decays exponentially, uniformly in the truncation. The implicit formula then gives the same conclusion for $t - Z$ and its derivatives.

The second equation can consequently be written, with an error exponentially small through every fixed number of derivatives, as

$$\begin{aligned} k\Delta_g Z + kB_N(Z)|dZ|_g^2 &= \rho - \vartheta(N(Z + \epsilon))C_N\rho_0 + O_j(e^{-a_N r}), \\ B_N(z) &= (k-1) + p(z) - \delta_0 s_{p(z)}. \end{aligned} \tag{4.32}$$

To verify this formula, set $f = 0$ in the coefficients. Then $D = 1$, $t = Z$, $w = 0$, $A_\chi = I$, $\mathcal{T} = ks_{p(Z)}|dZ|^2$, and $V = ku_0(Z)\nabla Z$, where $u_0(z) = e^{(k-1)z}l(z)$ is the coefficient u restricted to $f = 0$ and $(\log u_0)'(z) = k-1 + p(z)$. Every difference from these expressions has an exponentially small factor involving a derivative of f ; its other factors are bounded at fixed N by local regularity. This also proves the differentiated error assertion. The outer boundary causes no loss, since the same fixed-size local estimates hold there uniformly in R .

Define a smooth increasing function Q_N on the bounded range of Z by

$$Q_N(0) = 0, \quad Q'_N(z) = \exp\left(\int_0^z B_N(\tau) d\tau\right).$$

Its derivative has positive upper and lower bounds at fixed N . Equation (4.32) and the chain rule give

$$|\Delta_g Q_N(Z)| \leq C'_N r^{-n-\beta}, \quad Q_N(Z)|_{S_R} = 0. \tag{4.33}$$

Here C'_N denotes a constant at fixed N , not necessarily the penalty polynomial. The function

$$b_*(r) = r^{2-n}(1 - r^{-\beta/2})$$

is positive for $r > 1$ and satisfies

$$\Delta_g b_* = -\frac{\beta}{2} \left(n - 2 + \frac{\beta}{2}\right) r^{-n-\beta/2} + O(r^{2-2n}) < 0$$

sufficiently far out. The correction dominates the metric error because $\beta/2 < n-2$. Taking a sufficiently large multiple of b_* dominates both the source in Equation (4.33) and the inner

boundary values. Comparison with its two signs, also valid on the outer zero sphere, gives $|Q_N(Z)| \leq C'_N r^{2-n}$ on every truncation. Monotonicity of Q_N gives $Z = O(r^{2-n})$, and hence the same bound for t .

Rescale successive end annuli to unit size. The metric coefficients have uniformly controlled derivatives and converge to the Euclidean ones. The right side of the equation for $Q_N(Z)$ is a smooth function of Z times $r^{-n-\beta}$ plus exponentially small terms. Interior Poisson $W^{2,p}$ estimates, then Schauder estimates and differentiation, give $Z, t = O_j(r^{2-n})$ for each j ; the constants may depend on N, j . Finally $|dZ|^2 = O(r^{2-2n}) = O(r^{-n-\beta})$, since $\beta < n - 2$. Equation (4.32) now proves the stated $\Delta_g t$ bound and finishes the theorem. $\square$

For the resulting solution, Proposition 3.4 now applies with all of its end hypotheses verified. In particular, with $\omega = \omega_{n-1}$,

$$E(\widehat{g}) - E_0 = -\frac{1}{k\omega} \int_{\mathcal{M}_N} u \Xi dV_g \leq \mathcal{P}_N(\ell + \ell^{1/2}). \quad (4.34)$$

Here E_0 is the energy of the fixed input data. The metric also satisfies $\widehat{g} \geq e^{-2\epsilon} g$ pointwise. The flux estimate uses the polynomial penalty-band calculation of that proposition; the arbitrarily N -dependent constants in the end regularity estimates are used only to justify the flux and its normalization.

Remark 4.8. The construction exhausts the radii only after N has been fixed. Equations (4.1) and (4.2) are the estimates used with $N \rightarrow \infty$ in the next section. No bound for a high derivative that depends arbitrarily on N is substituted for a polynomial bound. In particular, this proof takes no elliptic limit as $N \rightarrow \infty$ and imposes no equation on a limiting singular hypersurface.

5 Height separation and a perimeter-minimizing enclosure

Throughout this section the data satisfy Definition 2.1. In particular, K has compact support, the dominant energy inequality is strict, and every component of S has strictly negative future expansion. Write a for the enclosing-area infimum of these data. Fix $0 < \epsilon < 1$. We use the smooth solution on the filled manifold $\mathcal{M}_N$ supplied by Theorem 4.1, with the notation of Proposition 3.1. Thus

$$\ell = e^{-N\epsilon}, \quad \eta = \ell^{3/2}, \quad h = \eta f, \quad \widehat{g} = e^{2t}(g + l^2 df \otimes df). \quad (5.1)$$

On the filling, g denotes its smooth extension. All comparisons with the original cut infimum will take place in the original exterior.

We first state precisely what is needed from the analytic construction. After increasing a polynomial $P_N = P(N)$, its estimates give

$$-\epsilon < t \leq Z \leq P_N, \quad \ell \leq l \leq e, \quad |f| + |df|_g \leq e^{P_N}, \quad |h| \leq C_0. \quad (5.2)$$

Moreover, $|h| \leq Cr^{-b_1}$ on the original end, with C independent of N . Local geometry of the background filling is uniformly bounded; its added product lengths and volume are $O(1 + N)$. In particular it has uniform coordinate balls of some radius $r_0 > 0$, with uniform local isoperimetric constants. For each fixed N , the end estimates are

$$t = O_j(r^{2-n}), \quad \Delta_g t = O(r^{-n-\beta}), \quad f \text{ and its derivatives decay exponentially,} \quad (5.3)$$

where $\beta > 0$. Constants in Equation (5.3) may depend arbitrarily on fixed N . Only Equation (5.2) and the uniform height barrier will be used in estimates required to be polynomial in N .

The height remains uniformly negative on the filling and tends uniformly to zero on the original end. A cutoff in this height therefore places a large constant conformal factor around the filling and leaves the metric unchanged near infinity, preserving the ADM energy. The weighted derivative estimates for h keep the scalar-curvature errors small where the factor varies. We minimize perimeter among sets containing a distance neighborhood of the region where $|h|$ is large; that neighborhood lies inside the constant-factor region. Contact with that obstacle would give positive volume densities on both sides of the frontier; the plateau would turn the resulting perimeter into an area larger than a distant competitor. Detachment places the exterior and a neighborhood of its frontier in the region of controlled scalar curvature. The construction and its estimates are at fixed input data and fixed ϵ .

To finish Theorem 2.2, choose the largest minimizing filled set from Lemma 5.3. Its complement is connected, and Lemma 5.4 puts its frontier Σ_N in the curvature region of Lemma 5.2. Smooth approximation by original enclosing cuts bounds its full g -area below by a . The comparison $\tilde{g}_N \geq e^{-2\epsilon}g$ then gives

$$\mathcal{H}_{g_N}^k(\Sigma_N) \geq e^{-k\epsilon}a.$$

Local regularity of perimeter-minimizing boundaries and completeness supply the remaining geometric assertions.

5.1 A negative height on the whole filling

Lemma 5.1 (The cap and collar barrier). *The product lengths in the filling can be chosen $O(1 + N)$ so that the final solution of Theorem 4.1 satisfies, for some $b_2 > 0$ independent of N and all sufficiently large integers N ,*

$$h < -b_2 \quad \text{on the whole added filling and a neighborhood of } S. \quad (5.4)$$

Here K is the fixed extended input tensor.

Proof. Use the signed transverse coordinate s increasing from each product neck toward S and the original end. The fixed transition from product geometry to an original collar has been chosen so that its leaves have

$$H_s + \text{tr}_{TS_s} K \leq -4\delta \quad (5.5)$$

for some fixed $\delta > 0$. Shrink δ when necessary. On the cap and the product part $K = -L_0g$, where L_0 is a fixed large constant.

It is useful to write the trace equation directly in terms of h . At a point where a test function has the same gradient as h , put $a_0 = \eta/l$ and let χ have its value for the solution there. The operator applied to the test function v_* is

$$\mathcal{Q}(v_*) = \text{tr}_{A_\chi} K + \frac{\text{tr}_{A_\chi} \text{Hess}_g v_*}{\sqrt{a_0^2 + |dv_*|_g^2}}. \quad (5.6)$$

The equation is $\mathcal{Q}(h) = h$. The coefficients in Equation (5.6) are evaluated at the solution; we do not replace its value of t by one computed from the barrier. Nevertheless,

$$0 < \chi \leq d = \frac{a_0^2}{a_0^2 + |dh|_g^2}, \quad \frac{\eta}{e} \leq a_0 \leq \frac{\eta}{\ell} = \ell^{1/2}. \quad (5.7)$$

These inequalities suffice for a comparison at a positive maximum of $h - v_*$.

Set $v_* = -\delta$ on the cap. On a product neck let $q_N = v'_* \geq 0$ start from zero, increase approximately exponentially, and end at a small fixed number $q_* > 0$. One explicit construction is to take an exponential $q_* e^{s-L_N}$ where its size lies between a constant multiple of η and q_* , smooth its end to the constant q_* on an interval of fixed length, and multiply its initial part by a fixed smooth cutoff on an interval where it is $O(\eta)$. Choose

$$L_N = \log(q_*/\eta) + O(1) = O(1 + N).$$

The cutoff can be flat at its initial endpoint. This construction gives

$$|q'_N| \leq C(q_N + \eta), \quad \int q_N ds \leq Cq_* + C\eta, \quad (5.8)$$

with C independent of N . Thus v_* glues smoothly to the constant on the cap. The same construction is made separately on every neck. The cap value is common to all components.

The product Hessian has only an axial component. Equations (5.7)–(5.8) give, even where the slope vanishes,

$$\frac{\chi|v''_*|}{\sqrt{a_0^2 + |v'_*|^2}} \leq C \frac{q_N + \eta}{\sqrt{(\eta/e)^2 + q_N^2}} \leq C'.$$

Consequently $\mathcal{Q}(v_*) \leq -kL_0 + C'$ on this part. Choose L_0 large enough that this is less than -2δ . Since $v_* \geq -\delta$, this is a strict supersolution inequality $\mathcal{Q}(v_*) < v_*$. It also holds on the constant cap, where the trace of K is $-nL_0$.

Continue with the fixed slope q_* across the fixed transition and up to S . Choose q_* so small that all the increase up to S , including Equation (5.8), is less than $\delta/4$. Beyond S , still inside the fixed strictly trapped collar, increase the slope smoothly until $v_* > C_0 + 1$ at the outer edge of that collar. This requires only a fixed, possibly large, slope and fixed derivatives; they do not depend on N . Keep $v'_* \geq q_*$ on the entire transition and collar. For such a function of signed distance,

$$\mathcal{Q}(v_*) = \text{tr}_{TS_s} K + \chi K(\partial_s, \partial_s) + \frac{v'_*}{\sqrt{a_0^2 + (v'_*)^2}} H_s + \frac{\chi v''_*}{\sqrt{a_0^2 + (v'_*)^2}}.$$

Equation (5.7) shows that this converges uniformly to the left side of Equation (5.5). For example, the absolute error is bounded by $C\ell/q_*^2 + C\ell \sup |v''_*|/q_*^3$, with fixed C . For large N it is therefore less than δ . As $v_* \geq -\delta$, again $\mathcal{Q}(v_*) < v_*$.

Apply the maximum test on the compact region consisting of the cap, necks, transitions, and these outer collars. On its outer boundary $h - v_* < 0$. At a positive interior maximum, the gradients agree and $\text{Hess } h \leq \text{Hess } v_*$. Positive definiteness of A_χ would give

$$h = \mathcal{Q}(h) \leq \mathcal{Q}(v_*) < v_* < h,$$

a contradiction. Thus $h \leq v_*$. Up to S the barrier is less than $-3\delta/4$. Continuity of the fixed collar barrier gives the same strict negative bound on a small neighborhood of S . Taking, for example, $b_2 = \delta/2$ proves the assertion. $\square$

5.2 Increasing area without losing exterior scalar curvature

Choose $0 < b_3 < b_2/4$ so small that

$$\mu - |J|_g - \rho - b_3|\tau| \geq \frac{1}{2}\rho > 0 \quad \text{on the original exterior.} \quad (5.9)$$

This follows from the strict energy margin in Equation (2.1) and the choice of ρ in the scalar construction: outside the compact support of τ no smallness condition on b_3 is needed, and on that compact set choose $b_3|\tau| \leq \rho/2$. Let $\zeta \in C^\infty(\mathbb{R})$ be nonnegative, equal to zero for $|z| \leq b_3/4$, equal to one for $|z| \geq b_3/2$, and with values in $[0, 1]$. Define

$$A_N = (\ell/\eta)^{1/2} = \ell^{-1/4}, \quad \psi_N = A_N \zeta(h), \quad \tilde{g}_N = e^{2\psi_N} \hat{g}. \quad (5.10)$$

The height barrier on the end puts $\{|h| \geq b_3/4\}$ in a fixed original coordinate radius, together with the added filling. Hence ψ_N vanishes identically near infinity. It changes neither the ADM energy nor the asymptotic estimates of $\hat{g}$.

Lemma 5.2 (Curvature on the small-height exterior). *For all sufficiently large N ,*

$$R_{\tilde{g}_N} \geq 0 \quad \text{on } \Omega \cap \{|h| \leq b_3\}.$$

On the original exterior the metric satisfies $\tilde{g}_N \geq e^{-2\epsilon} g$. Its ADM energy obeys

$$E(\tilde{g}_N) \leq E + P_N(\ell + \ell^{1/2}). \quad (5.11)$$

Proof. The solution supplied by Theorem 4.1 has $t > -\epsilon$ and the parameters of the scalar construction. Thus Equation (3.16) applies without a change of data or normalization. On $|h| \leq b_3$, its constraint and trace terms have lower bound $\rho/2$ by Equation (5.9); the remaining terms are nonnegative and retain the positive contribution $(1 - \delta_0)\mathcal{T}$.

For the new conformal factor we use Equation (3.18), with $\eta = \ell^{3/2}$. Its constants are polynomial in N for these fixed input data. The factor $\sqrt{d}$ in the height Laplacian cancels the axial loss in weighted coercivity, as shown immediately after that equation. No arbitrary fixed- N estimate for unweighted second derivatives enters this bound.

The conformal scalar-curvature law in dimension $n = k + 1$ reads

$$\frac{1}{2}e^{2(t+\psi_N)}R_{\tilde{g}_N} = \frac{1}{2}e^{2t}R_{\hat{g}} - ke^{2t}\Delta_{\hat{g}}\psi_N - \frac{k(k-1)}{2}e^{2t}|d\psi_N|_{\hat{g}}^2. \quad (5.12)$$

The chain rule, Equation (3.18), and the fixed derivative bounds of ζ imply

$$\begin{aligned} |e^{2t}\Delta_{\hat{g}}\psi_N| &\leq P_N\ell^{1/4}(1 + \sqrt{\mathcal{T}}) + C\ell^{3/4}, \\ e^{2t}|d\psi_N|_{\hat{g}}^2 &\leq C\ell^{1/2}. \end{aligned}$$

On the support of these errors $b_3/4 \leq |h| \leq b_3/2$. This region is disjoint from the filling by Lemma 5.1, and is contained in a fixed compact portion of the original exterior. There $\rho \geq c_* > 0$. Young's inequality bounds the error containing $\sqrt{\mathcal{T}}$ by $(1 - \delta_0)\mathcal{T}/2 + P_N\ell^{1/2}$. All remaining errors tend to zero. Equation (3.16) therefore proves nonnegative scalar curvature there for large N . Where ζ is constant, Equation (5.12) has no error terms and the same conclusion follows directly.

The floor $t > -\epsilon$, the nonnegative graph term, and $\psi_N \geq 0$ give the metric comparison. Finally $\psi_N = 0$ near infinity, so Equation (5.11) is exactly Proposition 3.4. In particular its sign is the upper energy estimate required for the Riemannian inequality. $\square$

5.3 Perimeter minimization and detachment from a distance obstacle

We use full perimeter in the filled manifold. For a measurable set F and an open set U , write $P_\gamma(F; U) = |D\mathbf{1}_F|_\gamma(U)$ for perimeter in a smooth metric γ , and $P_\gamma(F) = P_\gamma(F; \mathcal{M}_N)$. The

frontier is the support of this perimeter measure. For a set of finite perimeter its integration boundary is the reduced boundary ∂^*F , and $P_\gamma(F; U) = \mathcal{H}_\gamma^k(\partial^*F \cap U)$. For a measurable set F , $F^{(1)}$ denotes the points of volume density one; F^c denotes its complement in the filled manifold. These conventions distinguish full perimeter from a relative frontier inside Ω .

The final exterior and an ambient neighborhood of its frontier must lie in the original small-height region where Lemma 5.2 applies. We therefore enclose the whole set where $|h| \geq b_3$, which contains the filling, and use a thin distance neighborhood of that set as the obstacle. Its distance collar will supply the filled-side density if a minimizing frontier touches it. Put

$$C_N^h = \{|h| \geq b_3\}, \quad O_N = \{x : \text{dist}_g(x, C_N^h) \leq s_N\}, \quad s_N = e^{-Q_N}, \quad (5.13)$$

where Q_N is a sufficiently large polynomial. It can be chosen so that $s_N < r_0/10$ and

$$3s_N \sup |dh|_g < b_3/2. \quad (5.14)$$

Indeed $|dh| \leq \eta e^{P_N}$ by Equation (5.2). The entire $2s_N$ -neighborhood of O_N then lies in $\{|h| > b_3/2\}$, where $\tilde{g}_N = e^{2A_N} \hat{g}$. The compact obstacle O_N contains the whole added filling and a neighborhood of S . No smoothness of ∂O_N is asserted or needed. Outside O_N we are in $\text{int } \Omega \cap \{|h| < b_3\}$; detachment from O_N will provide the required ambient neighborhood as well.

Lemma 5.3 (Existence with a compact support bound). *There is a bounded finite-perimeter set $F_N \supset O_N$, modulo null sets, minimizing $P_{\tilde{g}_N}$ among all bounded sets containing O_N . One may choose it to contain every other minimizer modulo null sets. Its perimeter satisfies*

$$P_{\tilde{g}_N}(F_N) \leq e^{P_N}. \quad (5.15)$$

Proof. At fixed N the end is smooth and asymptotically Euclidean with the differentiated falloff in Equation (5.3). Consequently sufficiently large coordinate spheres have positive mean curvature for $\tilde{g}_N$ toward infinity. The required radius may depend arbitrarily on N .

Here is the clipping argument that gives the compact support needed for the direct method. All objects in this paragraph use $\tilde{g}_N$. On the large end let $Y = \nabla r / |\nabla r|$, so $|Y| = 1$ and $\text{div } Y = H_r > 0$. For a bounded finite-perimeter competitor F and almost every large R , Gauss–Green on $F \cap \{r > R\}$ gives

$$\int_{F \cap \{r > R\}} H_r dV = \int_{\partial^* F \cap \{r > R\}} \langle Y, \nu_F \rangle dA - \mathcal{H}^k(F^{(1)} \cap S_R).$$

The spherical normal here is $-Y$. Boundedness of F eliminates an outer flux; alternatively first cut off Y beyond its support. The first boundary integral is at most $P(F; \{r > R\})$. The BV truncation formula therefore yields the stronger clipping inequality

$$P_{\tilde{g}_N}(F \cap \{r < R\}) \leq P_{\tilde{g}_N}(F) - \int_{F \cap \{r > R\}} H_r dV_{\tilde{g}_N}. \quad (5.16)$$

The Gauss–Green and truncation formulas hold at almost every radius by the trace and slicing theorems for BV functions [3]. The notation $\{r < R\}$ here includes the entire compact core. In particular it contains O_N once R is sufficiently large.

Choose a fixed interval of such radii beyond the obstacle. Applying Equation (5.16) to each term of a minimizing sequence, with an almost-everywhere admissible radius in that interval, confines the sequence to one compact set. BV compactness gives L^1 convergence along a

subsequence, and lower semicontinuity gives a minimizer. Containment of O_N is preserved almost everywhere. The minimum is the unrestricted bounded-competitor minimum, so local variations are not subject to an artificial outer constraint.

Choose a fixed original coordinate sphere S_{R_*} so far out that $|h| < b_3/4$ on and beyond it for every N . It encloses O_N when N is large. On it $\psi_N = 0$, and Equation (5.2) implies $\hat{g} \leq e^{P_N} g$. Its filled side is a competitor; increasing P_N gives Equation (5.15). The fixed sphere need not lie in the region used for clipping: the latter was needed only for compactness at fixed N .

All minimizers have a common compact support bound: if a minimizer had positive volume beyond an almost-everywhere admissible clipping radius, the strictly positive last integral in Equation (5.16) would contradict minimality. Among minimizers, maximize the enclosed volume. Compactness, lower semicontinuity, and convergence of volume show that this maximum is attained. Perimeter submodularity implies

$$P(F \cup G) + P(F \cap G) \leq P(F) + P(G).$$

For two minimizers both sets on the left contain O_N , so equality holds and their union is a minimizer. Maximal volume therefore implies $G \subset F_N$ modulo null sets for every minimizer G . $\square$

Lemma 5.4 (Detachment from the obstacle). *The maximal minimizer F_N in Lemma 5.3 is detached from the obstacle O_N : for all sufficiently large N ,*

$$\text{supp } |D\mathbf{1}_{F_N}| \cap O_N = \emptyset. \quad (5.17)$$

Proof. The two different density arguments are illustrated in Figure 1. Suppose p belongs to this intersection. It cannot lie in the interior of O_N , because F_N contains O_N almost everywhere. Thus $\text{dist}_g(p, C_N^h) = s_N$. The complete background metric has a minimizing geodesic from some $z \in C_N^h$ to p . In particular $B_{s_N}^g(z) \subset O_N$. The midpoint of this geodesic is the center of a ball of radius $s_N/4$ contained in both O_N and $B_{s_N}^g(p)$. Uniform background ball-volume bounds give

$$\text{Vol}_g(F_N \cap B_{s_N}^g(p)) \geq cs_N^n. \quad (5.18)$$

This is the filled-side density, obtained from the distance obstacle.

We derive the other density from minimization. On $B_{s_N}^g(p)$ the high conformal multiplier is constant. Remove it and write $\gamma = \hat{g}$. Equation (5.2) gives, on this ball and every k -plane,

$$e^{-P_N} g \leq \gamma \leq e^{P_N} g, \quad e^{-P_N} dA_g \leq dA_\gamma \leq e^{P_N} dA_g, \quad (5.19)$$

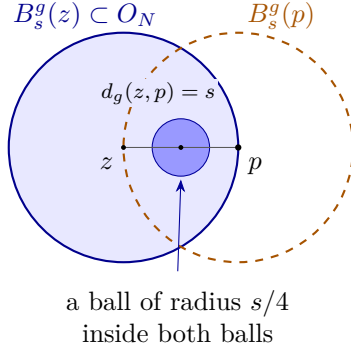
after changing the polynomial. No derivative of γ appears in this comparison.

Put $U = \mathcal{M}_N \setminus F_N$ and $V(r) = \text{Vol}_g(U \cap B_r^g(p))$. Since p is in the support of perimeter, $V(r) > 0$ for every $r > 0$: otherwise $\mathbf{1}_{F_N}$ would be constant almost everywhere on some neighborhood of p . For almost every $0 < r < s_N$, filling the ball is an admissible competitor. Cancel the unchanged exterior perimeter and the common constant factor e^{kA_N} . The perimeter truncation formula gives

$$P_\gamma(U; B_r^g(p)) \leq \mathcal{H}_\gamma^k(U^{(1)} \cap \partial B_r^g(p)).$$

Here $U^{(1)}$ is the measure-theoretic interior; the displayed trace agrees with it for almost every radius. The coarea formula in the smooth background metric and Equation (5.19) then yield

$$P_g(U; B_r^g(p)) \leq e^{P_N} V'(r) \quad \text{for almost every } r \in (0, s_N). \quad (5.20)$$



Obstacle inclusion gives
 $\text{Vol}_g(F_N \cap B_s^g(p)) \geq cs^n$.

Filling-ball comparison gives
 $\text{Vol}_g(F_N^c \cap B_s^g(p)) \geq e^{-P_N} s^n$.

Figure 1: The two densities at a hypothetical contact point $p \in \text{supp } |D\mathbf{1}_{F_N}| \cap \partial O_N$, with $z \in C_N^h$ chosen so that $d_g(z, p) = s = s_N$. The circles depict background metric balls schematically; no regularity of the obstacle or of the contact is assumed. Their overlap provides the first density. Minimality provides the second. Relative isoperimetry then forces perimeter inside the large constant conformal plateau, contradicting the competitor bound.

The absolute isoperimetric inequality on a uniformly controlled background ball, applied to the zero extension of $\mathbf{1}_{U \cap B_r^g(p)}$, includes its spherical trace:

$$V(r)^{(n-1)/n} \leq C(P_g(U; B_r^g(p)) + V'(r)) \leq e^{P_N} V'(r).$$

Local Euclidean isoperimetry in a uniform coordinate ball proves this form; the smooth background metric changes its constant by a fixed factor. Since V is absolutely continuous and positive for $r > 0$, its n th root satisfies

$$(V^{1/n})'(r) \geq n^{-1} e^{-P_N} \quad \text{almost everywhere on } (0, s_N).$$

Integrate from $a > 0$ to r and let $a \downarrow 0$. The result is

$$V(r) \geq e^{-P_N} r^n \quad (0 < r \leq s_N), \quad (5.21)$$

with another polynomial. This proves the unfilled-side density from a one-sided comparison, without assuming that the obstacle contact is a regular point or that the deformed metric has controlled derivatives.

Relative isoperimetry in $B_{s_N}^g(p)$, together with Equations (5.18) and (5.21), now gives

$$P_g(F_N; B_{s_N}^g(p)) \geq e^{-P_N} s_N^{n-1}.$$

Absorb $s_N^{n-1} = e^{-(n-1)Q_N}$ into the polynomial exponential, use Equation (5.19), and restore the constant multiplier. We obtain

$$P_{g_N}^-(F_N) \geq \exp(kA_N - P_N) = \exp(k\epsilon^{N\epsilon/4} - P_N). \quad (5.22)$$

For fixed $\epsilon > 0$, $e^{N\epsilon/4}$ dominates every polynomial in N . Equation (5.22) contradicts Equation (5.15) for large N . $\square$

5.4 The exterior, its full boundary, and its area

Proof of Theorem 2.2. Choose F_N as in Lemma 5.3. Its frontier is compact and, by Lemma 5.4, has positive distance from O_N . Every sufficiently small variation of the set near a frontier point remains admissible. Thus its boundary is locally perimeter minimizing in the smooth metric $\tilde{g}_N$.

To apply minimizing-current regularity, work in an oriented ambient coordinate ball W disjoint from O_N and put $T = \partial[[F_N]]$. For a same-boundary integral-current competitor T' , the compactly supported cycle $T' - T$ bounds an integral n -current in this coordinate ball, by the cone construction. A full-dimensional integral current has the form $[[v]]$ for an integer-valued BV function v . Hence $T' = \partial[[\mathbf{1}_{F_N} + v]]$. Truncating the integer-valued function to $[0, 1]$ gives $\mathbf{1}_G$, equal to $\mathbf{1}_{F_N}$ outside the variation. The BV chain rule gives

$$P_{g_N}^-(F_N; W) \leq P_{g_N}^-(G; W) \leq |D(\mathbf{1}_{F_N} + v)|_{g_N}^-(W) = \text{Mass}_{g_N}^-(T'; W).$$

The first inequality is the set minimization already established. Hence T is locally mass minimizing in the smooth ambient metric. One may realize that metric locally by a smooth isometric embedding; competitors and masses are then those in the ambient submanifold of [43, Chapter 7, Section 5, Theorem 5.8]. The theorem gives a smooth embedded minimal support outside a relatively closed singular set. Its hypersurface dimension is $k = n - 1$, so the singular bound is $k - 7 = n - 8$. Multiplicity one for the boundary of a set is recorded in [43, Chapter 7, Section 5, Remark 5.2]. The compactness and tangent-cone conclusions used in the minimizing-boundary regularity argument are Theorems 5.3 and 5.5 of that section.

We also specify the representative of the filled set. Since the frontier is detached from the obstacle, both insertion and deletion of a sufficiently small ball are admissible. The coarea and isoperimetric argument used in the preceding lemma, now applied to each side, gives positive lower volume densities for both F_N and its complement at every point in the support of perimeter. On a ball not meeting that support the indicator has zero distributional derivative, hence is constant almost everywhere. Let U_N be the union of those balls on which that constant is zero, and let the closed filled representative be $\mathcal{M}_N \setminus U_N$. The two density bounds identify its topological frontier with the perimeter support: every neighborhood of a support point contains both filled and unfilled points. Conversely a point outside the support has a constant-indicator neighborhood. This representative agrees with F_N up to a null set; the regularity theorem makes its frontier n -dimensional-volume null. Its perimeter and all the preceding minimizing comparisons are therefore unchanged.

Put $\Sigma_N = \partial U_N$. There is only one component of U_N . Indeed there is exactly one unbounded component, since F_N is bounded and the ambient manifold has just one end. Filling all bounded complementary components cannot increase perimeter. This statement can be seen directly on regular boundary charts, where it deletes the separating sheets and adds no sheet; the singular set has zero $\mathcal{H}^k$ measure. Equivalently it is the component decomposition and perimeter additivity for finite-perimeter sets [3]. The resulting set is still an admissible bounded competitor. If any bounded open component existed, its positive volume would strictly increase the filled volume; minimality would either be contradicted by a perimeter decrease or preserved with a volume increase. Both alternatives contradict the choice of F_N . Thus U_N is connected. We take $X_N = \overline{U_N}$, with its ambient boundary included.

The exterior lies in $\Omega \cap \{|h| < b_3\}$, because the filled set contains $O_N \supset C_N^h$. In fact it has a smooth ambient neighborhood in that region: detachment gives positive separation between Σ_N and O_N , and compactness permits a smooth compact domain containing O_N whose boundary is still strictly inside the filled set. Remove that domain. On the remaining smooth ambient

exterior, Lemma 5.2 gives nonnegative scalar curvature, also on a neighborhood of Σ_N .

All local variations of Σ_N have zero first variation. More precisely, if Y is a compactly supported smooth ambient vector field near the frontier and Φ_s is its flow, then $\Phi_s(F_N)$ remains admissible for both signs of small s . Differentiation of the perimeter area formula gives

$$\int_{\partial^* F_N} \operatorname{div}_{T_x \partial^* F_N} Y d\mathcal{H}_{g_N}^k = 0. \quad (5.23)$$

This is stationarity of the full boundary varifold, including the assertion that there is no additional first-variation term at the singular set. Global outer minimization follows because every bounded superset of F_N also contains O_N . The maximal choice of F_N also makes it the outermost member among enclosures with this same minimum perimeter, although outermostness is not required below.

We next compare the full frontier with original enclosing hypersurfaces. Detachment and Lemma 5.1 put Σ_N in $\operatorname{int} \Omega$, separated from a neighborhood of the entire original filling, including every component of S . There are bounded smooth open sets F_j in the filled manifold such that

$$\mathbf{1}_{F_j} \longrightarrow \mathbf{1}_{F_N} \text{ in } L^1, \quad P_g(F_j) \longrightarrow P_g(F_N), \quad (5.24)$$

and each F_j still contains a fixed neighborhood of the whole filling. Here is a way to impose the last condition in strict BV approximation. Choose a smaller compact neighborhood on which $\mathbf{1}_{F_N} = 1$ almost everywhere, separated from Σ_N , and a compact exterior region where it is zero. Smooth the characteristic function in finitely many coordinate charts only in the remaining region, using a partition of unity and leaving those two constant regions fixed. The resulting smooth functions $u_j \in [0, 1]$ converge in L^1 with total variation converging to $P_g(F_N)$. Choose $\delta_j \downarrow 0$ so slowly that $\|u_j - \mathbf{1}_{F_N}\|_{L^1}/\delta_j \rightarrow 0$. By coarea there is a regular value $t_j \in (\delta_j, 1 - \delta_j)$ for which

$$P_g(\{u_j > t_j\}) \leq \frac{\int |du_j|_g dV_g}{1 - 2\delta_j} + o(1) = P_g(F_N) + o(1).$$

The L^1 difference of this level set from F_N is at most $\|u_j - \mathbf{1}_{F_N}\|_{L^1}/\delta_j$, and hence tends to zero. Lower semicontinuity gives the reverse perimeter $\liminf$. Thus $F_j = \{u_j > t_j\}$ satisfies Equation (5.24). This is the strict approximation/coarea construction of [3], localized away from the two constant regions.

For each j retain the component of $\mathcal{M}_N \setminus \overline{F_j}$ that contains the distant end, and let D_j be its closure. A compact smooth boundary has finitely many components; retaining this exterior component merely selects some of those entire boundary components. It preserves smoothness and can only decrease the full boundary area. Because F_j contains a neighborhood of the whole filling, $D_j \subset \Omega$, its interior lies in $\operatorname{int} \Omega$, and D_j is closed and connected with a smooth compact intrinsic boundary Γ_j . It is an admissible enclosing hypersurface in Definition 1.2. Thus

$$a \leq \operatorname{Area}_g(\Gamma_j) \leq P_g(F_j).$$

Taking the limit gives $P_g(F_N) \geq a$. No component of a cut has been discarded in evaluating its intrinsic boundary; the passage to D_j is an admissible choice of the exterior domain itself. There is no coincidence with S in this approximation because of the fixed neighborhood, and the original definition still allows coincidence for its other competitors.

Since $\tilde{g}_N \geq e^{-2\epsilon}g$ on Ω , comparison on every approximate tangent k -plane now yields

$$\mathcal{H}_{g_N}^k(\Sigma_N) = P_{g_N}^-(F_N) \geq e^{-k\epsilon} P_g(F_N) \geq e^{-k\epsilon} a. \quad (5.25)$$

The first equality uses the regularity theorem and the zero $\mathcal{H}^k$ measure of the singular set.

It remains to check completeness and the end. The filled metric is complete, since $\tilde{g}_N \geq e^{-2\epsilon} g$ and the background filled metric is complete. Its restriction to X_N is complete with the frontier included. Here completeness is understood in the convention for closed exteriors specified with the Riemannian input: the intrinsic length distance on X_N may be extended-valued, and its restriction to each finite-distance equivalence class is complete. For the intrinsic-length formulation, a Cauchy sequence has a subsequence whose successive intrinsic distances are less than 2^{-j} . Join successive terms by curves of length less than 2^{1-j} in X_N . Their concatenation has an ambient endpoint, by ambient completeness, and that endpoint belongs to X_N because X_N is closed. Appending this endpoint gives a curve in X_N whose tail length tends to zero, proving intrinsic convergence of the subsequence and then of the original Cauchy sequence. The frontier is included as part of the closed exterior used in the Riemannian input in Section 2.

Finally the end satisfies $\tilde{g}_N - I = O_j(r^{2-n})$. The conformal law applied to Equation (5.3) shows

$$R_{\tilde{g}_N} = O(r^{-n-\beta}) + O(r^{-2n+2}).$$

If needed decrease β so that $0 < \beta < n - 2$; then the displayed bound is $O(r^{-n-\beta})$. All graph errors decay exponentially, and ψ_N is zero on the end. Scalar curvature is integrable there and, being smooth and bounded on the compact ambient neighborhood of the frontier, integrable on the remainder of X_N . The energy estimate is Lemma 5.2. This proves all the stated properties. $\square$

Remark 5.5 (Flux and weak boundary conventions). The scalar flux in Proposition 3.4 is computed on the complete smooth filled manifold before an enclosure is chosen:

$$E(\hat{g}) - E = -\frac{1}{k\omega} \lim_{R \rightarrow \infty} \int_{S_R} g(V, \nu_g) dA_g = -\frac{1}{k\omega} \int_{\mathcal{M}_N} u \Xi dV_g.$$

Thus no elliptic boundary condition or unaccounted singular-boundary flux is involved. If one restricts a smooth vector field to X_N , the Gauss–Green theorem for finite-perimeter sets gives its ordinary reduced-boundary flux. For instance, let B_R denote the filled core together with the end region $r < R$, and let ν_R be the outward $\tilde{g}_N$ -unit normal to S_R . At a regular truncation radius,

$$\int_{U_N \cap B_R} \operatorname{div}_{\tilde{g}_N} Y dV_{\tilde{g}_N} = \int_{S_R} \tilde{g}_N(Y, \nu_R) dA_{\tilde{g}_N} - \int_{\partial^* F_N} \tilde{g}_N(Y, \nu_{F_N}) dA_{\tilde{g}_N},$$

when Y is supported in the indicated truncation or has its stated outer trace. The normal ν_{F_N} points from the filled set toward the end. There is no further measure on the singular set. This flux convention and Equation (5.23) are the weak identities attached to the constructed boundary.

6 Preparing an end with zero momentum

Put $k = n - 1$ and $p = n - 2 = k - 1$. A radius function ϱ below is smooth and positive on the closed exterior and equals the Euclidean radius on a sufficiently distant part of its coordinate end. Constants in the end-preparation argument may depend on the original data and on its fixed future-timelike ADM vector. In particular, no uniform estimate as $E - |P| \downarrow 0$ is asserted. Section 6.6 supplies the conformal energy increase used in Section 2 to prove automatic timelikeness; that construction does not assume timelikeness.

The target class is Definition 2.1. The present task is to reach it while comparing every enclosing hypersurface with an original cut.

Proposition 6.1 (End replacement and strict inequalities). *Let (Ω, g, K) be a smooth one-ended exterior as in Definition 1.1, with compact complement of its coordinate end and with*

$$g - \delta = O_6(r^{-q}), \quad K = O_5(r^{-1-q}), \quad \frac{n-2}{2} < q < n-2.$$

Assume integrable constraint densities satisfying $\mu \geq |J|_g$, finite ADM limits satisfying $E > |P|$, and $\theta_+(S) \leq 0$ on every boundary component. Suppose that the enclosing-area infimum of Definition 1.2 is $a = A_{\min, g}(S) > 0$, and put $m = (E^2 - |P|^2)^{1/2}$. There are data (g_j, K_j) satisfying Definition 2.1 on the same exterior, with one fixed $\beta \in (0, 1)$, numbers $\epsilon_j \downarrow 0$, and proper diffeomorphisms $\Phi_j : \Omega \rightarrow \Omega$ equal to the identity near S , such that

$$P_j = 0, \quad E_j \rightarrow m, \quad \Phi_j^* g \leq (1 + \epsilon_j)g_j, \quad A_{\min, g_j}(S) \geq (1 + \epsilon_j)^{-k/2}a. \quad (6.1)$$

The positive constant c_j in Equation (2.1) and the end coordinates may depend on j .

We first match the original energy and momentum with a boosted vacuum end. A compact annular correction joins the two ends with a constraint error whose total scale tends to zero. Within the vacuum region we then bend the slice to a static one, giving zero momentum. A positive conformal factor repairs the small annular error without decreasing the metric. A proper radial comparison map protects every enclosing hypersurface during the change of slice. For each already fixed replacement, a second scalar perturbation makes the energy condition and future outer trapping strict while changing the energy by a quantity tending to zero. The argument fixes the original boost throughout; it needs no estimates uniform as the ADM vector approaches the null cone.

An asymptotic change to a rest frame is also used by Allen, Bryden, Kazaras and Khuri in the invariant-mass step of [2, Section 5]. Here the end is first replaced by an explicit vacuum model, and the proper comparison map in Proposition 6.1 controls every enclosing hypersurface during the replacement and bending.

6.1 Conformal changes and a vacuum model with matching charges

The conformal identities below explain how a scalar function can improve the constraint inequality and the boundary expansion at the same time. We then compute the vacuum model's charges directly in the ADM normalization used here.

For a smooth function s set $g_s = e^{2s}g$ and $K_s = e^s K$. If $|v|_g \leq 1$, the corresponding vector in the new unit ball is $v_s = e^{-s}v$. The conformal connection formula and scalar curvature formula give

$$\begin{aligned} e^{2s}(\mu_s + J_s(v_s)) &= \mu + J(v) - k\Delta_g s - \frac{kp}{2}|ds|_g^2 + kK(\nabla s, v), \\ e^s \theta_{+, s} &= \theta_+ + k\partial_\nu s. \end{aligned} \quad (6.2)$$

Here the normal ν points into the exterior. For completeness, $\tau_s = e^{-s}\tau$, $|K_s|_{g_s}^2 = e^{-2s}|K|_g^2$, and

$$R_{g_s} = e^{-2s}(R_g - 2k\Delta_g s - kp|ds|_g^2).$$

Writing $\pi = K - \tau g$, contraction of $\Gamma(g_s)_{ij}^l - \Gamma(g)_{ij}^l = s_i \delta_j^l + s_j \delta_i^l - g_{ij} s^l$ yields $J_{s,i} = e^{-s}(J_i + kK_{ij}\nabla^j s)$. The tangential trace of K scales by e^{-s} and $H_s = e^{-s}(H + k\partial_\nu s)$, proving both identities. In particular, if $s = 2p^{-1} \log U$, $U > 0$, the entire added term in the first identity is

$$\frac{2k}{pU}(-\Delta_g U + K(\nabla U, v)). \quad (6.3)$$

Lemma 6.2 (Charges of a boosted Schwarzschild–Tangherlini end). *In the Schwarzschild–Tangherlini spacetime of geometric mass $M > 0$, let b, y be static time and isotropic spatial coordinates. Make the Lorentz change*

$$b = \gamma(T + vx^1), \quad y^1 = \gamma(x^1 + vT), \quad y^a = x^a \quad (2 \leq a \leq n), \quad \gamma = (1 - v^2)^{-1/2}, \quad (6.4)$$

where $0 \leq v < 1$. The induced data on $T = 0$, oriented by the future normal, have

$$E = \gamma M, \quad P_1 = \gamma Mv, \quad P_a = 0 \quad (a > 1). \quad (6.5)$$

In these coordinates the induced fields satisfy $g - \delta = O_d(r^{-p})$ and $K = O_d(r^{-p-1})$ for every fixed nonnegative integer d , where $p = n - 2$.

Proof. We compute the homogeneous leading metric and second-form terms, with remainders $O_d(r^{-2p})$ and $O_d(r^{-2p-1})$, respectively. The exact isotropic metric is

$$-\left(\frac{1 - M/(2|y|^p)}{1 + M/(2|y|^p)}\right)^2 db^2 + \left(1 + \frac{M}{2|y|^p}\right)^{4/p} |dy|^2.$$

Its leading perturbation of Minkowski space is $2M|y|^{-p}(db^2 + p^{-1}|dy|^2)$. On $T = 0$ put

$$\rho^2 = \gamma^2(x^1)^2 + \sum_{a=2}^n (x^a)^2, \quad f = \rho^{-p}, \quad A = \gamma^2(v^2 + p^{-1}), \quad B = p^{-1}, \quad C = kp^{-1}\gamma^2v.$$

The nonzero relevant leading components of the spacetime perturbation $\mathbf{H}$ are $\mathbf{H}_{11} = 2MAf$, $\mathbf{H}_{aa} = 2MBf$, and $\mathbf{H}_{01} = 2MCf$. Moreover

$$\partial_1 f = -p\gamma^2 x^1 \rho^{-n}, \quad \partial_a f = -px^a \rho^{-n}, \quad \partial_T f = -p\gamma^2 vx^1 \rho^{-n}.$$

Thus the energy flux vector $Q_i = \partial_j \mathbf{H}_{ij} - \partial_i \mathbf{H}_{jj}$ satisfies

$$Q_1 = -2MkB\partial_1 f = 2Mk\gamma^2 x^1 \rho^{-n}, \quad Q_a = -2M(A + pB)\partial_a f = 2Mp(A + 1)x^a \rho^{-n}.$$

Since $p(A + 1) = k\gamma^2$, these expressions give the pointwise identity

$$Q_i = 2Mk\gamma^2 x_i \rho^{-n}. \quad (6.6)$$

The chosen sign of the second fundamental form gives, to first order,

$$K_{ij} = \frac{1}{2}(\partial_T \mathbf{H}_{ij} - \partial_i \mathbf{H}_{0j} - \partial_j \mathbf{H}_{0i}).$$

Direct substitution gives

$$\begin{aligned} K_{11} &= M\gamma^4 v(2p + 1 - pv^2)x^1 \rho^{-n}, & K_{aa} &= -M\gamma^2 vx^1 \rho^{-n}, \\ K_{1a} &= Mk\gamma^2 vx^a \rho^{-n}, & K_{ab} &= 0 \quad (a \neq b). \end{aligned}$$

Their Euclidean trace is $M\gamma^4 v(p + v^2)x^1 \rho^{-n}$. Consequently, for the leading Euclidean trace reversal $\pi_{ij} = K_{ij} - (\text{tr}_\delta K)\delta_{ij}$,

$$\pi_{1j} = Mk\gamma^2 vx_j \rho^{-n}, \quad \pi_{a1} = Mk\gamma^2 vx^a \rho^{-n}, \quad \pi_{ab} = -Mk\gamma^4 vx^1 \rho^{-n} \delta_{ab}. \quad (6.7)$$

Using the exact normal and the exact metric in the trace reversal changes these formulas by $O_d(r^{-2p-1})$. The energy remainder has first derivatives of this same order. Their sphere integrals are $O(r^{-p})$, and therefore vanish for every $n \geq 3$.

The ellipsoid $\{|\text{diag}(\gamma, 1, \dots, 1)x| < 1\}$ has volume $\omega/(n\gamma)$. Polar integration therefore gives

$$\int_{S^{n-1}} |\text{diag}(\gamma, 1, \dots, 1)\theta|^{-n} d\theta = \frac{\omega}{\gamma}. \quad (6.8)$$

Equations (6.6) and (6.7), divided by $2k\omega$ and $k\omega$, give $E = M\gamma$ and $P_1 = M\gamma v$. The integrand of a transverse momentum row is a constant times $x^1 x^a \rho^{-n}/r$, whose integral vanishes by reflection. $\square$

For the original charges we now fix $M = m$, $v = |P|/E$, and rotate the first axis to the direction of P when $P \neq 0$. Lemma 6.2 gives a model end (g^b, K^b) with exactly the same charges. The parameter v is fixed in all limits that follow.

6.2 Joining the original and model ends

A direct cutoff need not preserve the dominant energy condition. Our immediate task is to remove its leading constraint error by compactly supported corrections. The obstructions to a compact correction are finitely many moments. Charge matching removes their constant limits, and integrability makes the remaining first moments small enough after rescaling.

Localized constraint deformation and the finite-dimensional obstruction from the adjoint kernel are central to the gluing methods of Corvino–Schoen [15] and Chruściel–Delay [14]. We use this methodological starting point, not an imported rest-end replacement theorem: the present data need not be vacuum, no parity is imposed, and both the energy inequality and every enclosing hypersurface must survive the replacement.

We give the compact correction explicitly because neither parity conditions nor convergence of center of mass or angular momentum is assumed. On Euclidean tensors define

$$S(h) = \partial_i \partial_j (h_{ij} - (\text{tr}_\delta h) \delta_{ij}), \quad D(b)_i = \partial_j (b_{ij} - (\text{tr}_\delta b) \delta_{ij}).$$

Their formal adjoint kernels relevant to compact support are, respectively, affine scalar functions and Euclidean Killing fields.

Lemma 6.3 (Right inverses with compact support). *Let $\mathcal{A} \subset \mathbb{R}^n$ be a bounded connected Euclidean annulus and fix a compact subset of its interior. Smooth scalar sources supported in that subset with zero constant and first moments have smooth symmetric solutions of $S(h) = f$ supported in a larger fixed compact subset of $\mathcal{A}$. Smooth vector sources with zero integrals against every Euclidean Killing field have similarly supported symmetric solutions of $D(b) = F$. For $1 < a < \infty$ and integers $d \geq 0$, these solutions may be chosen linearly with estimates*

$$\|h\|_{W^{d+2,a}} \leq C_{d,a} \|f\|_{W^{d,a}}, \quad \|b\|_{W^{d+1,a}} \leq C_{d,a} \|F\|_{W^{d,a}}.$$

For arbitrary sources the same conclusions hold after subtracting fixed smooth moment bumps, whose coefficients are bounded by the absolute values of the respective moments.

Proof. We use the scalar divergence inverse with compactly supported output on a connected open set: a mean-zero smooth function supported in a fixed compact set has a compactly supported vector primitive, with one Sobolev derivative gained; the support and Sobolev mapping statements

are given, for example, by the Bogovskii operator in [16, Section 3.1, Equation (3.13), Theorem 3.2, Corollaries 3.3–3.4, Remark 3.5, and Proposition 4.1(ii)]. Writing its smooth unit-integral bump as η , the source's top-degree identity is $dT_n\omega = \omega - (\int \omega)\eta dz^1 \wedge \cdots \wedge dz^n$. For $\omega = f dz^1 \wedge \cdots \wedge dz^n$, identify the $(n-1)$ -form $T_n\omega$ with the vector field V by $T_n\omega = \iota_V(dz^1 \wedge \cdots \wedge dz^n)$. Then $dT_n\omega = (\operatorname{div} V) dz^1 \wedge \cdots \wedge dz^n$; the zero integral removes exactly the top-degree correction. The source gives this operator on a domain starlike with respect to the supporting ball. We obtain the annular operator by the following finite localization, preserving its support and derivative gain.

On a ball, choose a smooth function η of integral one supported in a smaller concentric ball and use the Bogovskii integral

$$(\mathcal{B}f)(x) = \int f(y)(x-y) \int_1^\infty \eta(y+t(x-y))t^{n-1} dt dy.$$

Differentiation in distributions gives $\operatorname{div} \mathcal{B}f = f - \eta \int f = f$; the segments occurring in the integral remain in the ball. For compact input support and compact $\operatorname{supp} \eta$, their union is compact in that ball. The kernel is of order $1-n$, and its differentiated kernel has the cancellation of a Calderon–Zygmund kernel. The resulting $W^{d,a}$ estimate is the usual boundedness of that singular integral and its commutators with derivatives. Cover the prescribed support by finitely many balls compact in $\mathcal{A}$, connect them by a finite chain of overlapping such balls, and use a partition of unity. Transfer the means along the chain using unit-integral bumps in the overlaps. Each resulting ball source has mean zero; the sum of its ball primitives proves the stated inverse on $\mathcal{A}$, with fixed support and constants.

Write $z = (z_1, \dots, z_n)$ for the Euclidean coordinates. Apply it first to a scalar f with its affine moments zero, obtaining $\partial_i v_i = f$. Compact support gives $\int v_i = -\int z_i f = 0$. Apply the same inverse to each v_i to obtain $\partial_j T_{ij} = v_i$. Then $Q = (T + T^t)/2$ has $\partial_i \partial_j Q_{ij} = f$. The tensor $h = Q - k^{-1}(\operatorname{tr}_\delta Q)\delta$ has trace reversal Q , so $S(h) = f$ with the asserted two-derivative gain.

For a vector source F , first solve $\partial_j B_{ij} = F_i$ row by row. Translation moments permit this solve. Rotation moments give

$$0 = \int (z_i F_j - z_j F_i) = \int (B_{ij} - B_{ji}).$$

Put $A_{ij} = (B_{ij} - B_{ji})/2$ and solve $\partial_l D_{ijl} = -A_{ij}$, choosing $D_{ijl} = -D_{jil}$. Set

$$E_{ij} = \partial_l (D_{ijl} - D_{ilj} - D_{jli}), \quad C_{ij} = B_{ij} + E_{ij}.$$

Index interchange gives $E_{ij} - E_{ji} = -2A_{ij}$ and $\partial_j E_{ij} = 0$: the first two double derivatives cancel after interchanging j, l , and the last vanishes by skew symmetry in those indices. Thus C is symmetric and $\partial_j C_{ij} = F_i$. Taking $b = C - k^{-1}(\operatorname{tr}_\delta C)\delta$ proves the assertion, including the derivative count.

Finally take a nonnegative smooth bump supported in an open ball of $\mathcal{A}$. The Gram matrix of the affine functions, or of the vector Killing fields, with this weight is positive definite: a polynomial in either finite-dimensional space vanishing on that ball vanishes identically. Multiplying the basis elements by the bump and the inverse Gram matrix produces fixed dual moment bumps. Subtraction removes all moments with the stated bound on coefficients. $\square$

The preceding inverses reduce the joining problem to estimates for the moments that they cannot remove. We apply them on the fixed annulus $\mathcal{A} = \{z \in \mathbb{R}^n : 1 < |z| < 4\}$. Choose a smooth cutoff $0 \leq \chi \leq 1$, equal to one near its inner sphere and zero near its outer sphere. For L large put $\chi_L(x) = \chi(x/L)$ on $\mathcal{A}_L = \{L < |x| < 4L\}$. The following observation explains why no center-of-mass or angular-momentum limit is required.

Lemma 6.4 (Sublinear first moments). *For the original data and the charge-matched reference end fixed above, let $h = g - \delta$, $h^b = g^b - \delta$, and $b = K$, $b^b = K^b$ on their common coordinate tail. Put $H = h - h^b$ and $B = b - b^b$. Define the scalar and vector commutators on $\mathcal{A}_L$ by*

$$C_L = S(\chi_L H) - \chi_L S(H), \quad F_L = D(\chi_L B) - \chi_L D(B),$$

and their rescaled versions on $\mathcal{A}$ by

$$\widehat{C}_L(z) = L^2 C_L(Lz), \quad \widehat{F}_L(z) = L^2 F_L(Lz).$$

The flat sources $S(H)$ and $D(B)$ are integrable. For every fixed affine scalar function $a(z)$ and every fixed Euclidean Killing field $Y(z)$,

$$\int_{\mathcal{A}} a \widehat{C}_L dz = o(L^{2-n}), \quad \int_{\mathcal{A}} Y^i \widehat{F}_{L,i} dz = o(L^{2-n}).$$

Proof. Expansion of the constraint formulas about $(\delta, 0)$ gives

$$S(h) = 2\mu + O(r^{-2q-2}), \quad D(K) = J + O(r^{-2q-2}). \quad (6.9)$$

The remainder estimates follow, respectively, from hD^2h , $(Dh)^2$, and K^2 , and from hDK and DhK . The volume densities of g and δ are uniformly comparable on the tail. Hence integrability of μ, J , together with $2q > n - 2$, gives the claim for the original data. The model is vacuum with faster decay, so its flat sources are integrable too.

For any integrable tail function f , even without an integrable first absolute moment,

$$\int_{R_0 < r < 4L} r |f| dx = o(L). \quad (6.10)$$

Indeed divide by L and split at a fixed large radius R_1 . The inner integral divided by L tends to zero, whereas the outer integral divided by L is at most $4 \int_{r > R_1} |f|$, which tends to zero as $R_1 \rightarrow \infty$.

Write $Q = H - (\text{tr}_\delta H)\delta$. For an affine function $a(x)$ define its Green flux by

$$\mathcal{B}_a(R) = \int_{S_R} (a \partial_j Q_{ij} \nu_\delta^i - (\partial_i a) Q_{ij} \nu_\delta^j) dA_\delta.$$

Integration by parts between spheres yields $\mathcal{B}_a(R) - \mathcal{B}_a(R_0) = \int_{R_0 < r < R} a S(H)$. Charge matching gives $\mathcal{B}_1(R) \rightarrow 0$, and Equation (6.10) gives $\mathcal{B}_{x^a}(R) = o(R)$. For the scalar commutator C_L , supported where the cutoff varies, one consequently has

$$\int_{L < r < 4L} a C_L = -\mathcal{B}_a(L) - \int_{L < r < 4L} \chi_L a S(H). \quad (6.11)$$

This is $o(1)$ for $a = 1$ and $o(L)$ for $a = x^a$.

For momentum put $Q_{ij} = (K - K^b)_{ij} - \text{tr}_\delta(K - K^b)\delta_{ij}$. For a Euclidean Killing field Y , symmetry of Q gives $Q_{ij}\partial_j Y_i = 0$. Its Green flux is thus simply $\int_{S_R} Y_i Q_{ij} \nu_\delta^j$. For constant Y its limit is zero by equality of the momenta; replacing geometric trace reversal by Euclidean trace reversal changes the sphere flux by $O(R^{n-2-2q}) \rightarrow 0$. For rotational Y the same Green identity and Equation (6.10) give $o(R)$. Equation (6.11) holds for F_L with this flux and pairing.

Under $x = Lz$ the fields are $h(Lz)$ and $LK(Lz)$, and both flat sources are multiplied by L^2 . A constant moment is therefore multiplied by L^{2-n} and a first moment by L^{1-n} . The preceding $o(1)$ and $o(L)$ bounds prove the same $o(L^{2-n})$ bound for all rescaled moments. $\square$

We now apply these two lemmas to the actual joining problem. Under $x = Lz$, interpolate the scaled fields on $\mathcal{A}$ between the original and model fields using χ . The errors in $(2\mu, J)$ relative to the cutoff interpolation of the exact constraint densities equal $\widehat{C}_L$ and $\widehat{F}_L$, respectively, plus $O(L^{-2q})$ in scaled units. The commutators are supported in the fixed compact subset where the cutoff varies. By Lemma 6.3 subtract compact corrections that cancel these commutators after their moment bumps have been removed. The commutators have $W^{1,a}$ norm $O(L^{-q})$, for any fixed $a > n$, because the available asymptotic derivatives include those of order three for g and two for K . Sobolev embedding in dimension n therefore gives corrections of size $O(L^{-q})$ in $C^{2,\alpha}$ for the scaled metric and in $C^{1,\alpha}$ for the scaled second fundamental form, where $0 < \alpha < 1 - n/a$.

These are smooth compact corrections; only their first few norms need be bounded uniformly in L . Their nonlinear contribution is still $O(L^{-2q})$. The fixed smooth moment bumps contribute $o(L^{2-n})$ by Lemma 6.4. Thus, returning to physical units, the new data $(\tilde{g}_L, \tilde{K}_L)$ agree with the original for $r \leq L$ and with the model for $r \geq 4L$, and satisfy

$$\tilde{\mu}_L - |\tilde{J}_L|_{\tilde{g}_L} \geq -d_L L^{-n} \mathbf{1}_{\{L < r < 4L\}}, \quad d_L \rightarrow 0. \quad (6.12)$$

To check the unit balls in this assertion, the interpolated metric, the original metric, and δ differ by $O(L^{-q})$. The scaled original J is $O(L^{-q})$. Transporting a unit vector between these balls thus changes its pairing with the interpolated J by $O(L^{-2q})$. The cutoff is between zero and one, so the original DEC and the vacuum model give the claimed sign after precisely the errors already estimated.

6.3 Changing the vacuum slice and repairing the constraints

We have joined the original data to a vacuum end with matching charges. The only possible failure of the dominant energy condition has the small bound in Equation (6.12). The next step removes the momentum by changing the slice entirely within that vacuum end; it therefore creates no further constraint error.

In the model coordinates, $T = 0$ is the graph $b = vy^1$. Choose a smooth cutoff ζ equal to one initially and zero finally, with

$$\left| \frac{d\zeta}{d \log |y|} \right| \leq \eta_v, \quad v(1 + \eta_v) < 1.$$

This can be done by taking a sufficiently long, but fixed, interval in $\log(|y|/L)$. Start it outside $|y| = 4\gamma L$, where the joined data are exactly the model. Replace the graph by $b = vy^1 \zeta(\log(|y|/L))$ there. Its Euclidean slope is at most $v(1 + \eta_v) < 1$; the Schwarzschild coefficients tend uniformly to their flat values on the transition as $L \rightarrow \infty$. Thus this is a smooth spacelike hypersurface for large L , and its induced data are exactly vacuum throughout the bending region. They agree with $b = 0$ beyond $|y| = C_v L$, for a fixed C_v . For $v = 0$ no bending is necessary.

Use $y = \text{diag}(\gamma, 1, \dots, 1)x$ to parametrize the whole construction outside the unchanged core. The resulting metric and second fundamental form, denoted $(\bar{g}_L, \bar{K}_L)$, satisfy, in this region,

$$c_v \delta \leq \bar{g}_L \leq C'_v \delta, \quad |D\bar{g}_L| + |\bar{K}_L| \leq C'_v/r. \quad (6.13)$$

On the joining annulus the stronger small perturbation estimates hold; in the bending region these bounds follow by differentiating the graph and its normal. The tensor $\bar{K}_L$ is compactly supported, and the only possible DEC deficit is Equation (6.12).

The slice is now static near infinity and its second fundamental form has compact support. It remains to repair the annular deficit. By Equation (6.3), it is enough to build a positive function

whose negative Laplacian dominates the drift $|\bar{K}_L| |dU_L|$ and the deficit. The function must be constant on the original core, so the boundary expansion keeps its sign.

We construct a nonnegative radial function $f_L(s)$, $s = |y|/L$, with bounded derivatives on every fixed transition interval, so that

$$U_L = 1 + e_L L^{-p} f_L(|y|/L), \quad e_L \downarrow 0, \quad (6.14)$$

repairs that deficit. Choose a smooth nonnegative χ_* equal to one on the radial interval containing the full joining annulus and supported in a slightly larger interval. Begin before that interval with $z_L = 0$, and solve

$$z'_L = C_* z_L + \chi_*, \quad z_L = -f'_L.$$

Continue this equation through the bending interval. The interval and C_* are fixed for the chosen boost. Thus z_L and its derivatives are bounded independently of large L . The radial function has $|ds|_{\bar{g}_L}^2 \geq cL^{-2}$, $|\Delta_{\bar{g}_L} s| \leq CL^{-2}$, and $|\bar{K}_L| |ds|_{\bar{g}_L} \leq CL^{-2}$ there, by Equation (6.13). Consequently

$$\begin{aligned} -\Delta_{\bar{g}_L} U_L - |\bar{K}_L| |dU_L|_{\bar{g}_L} &= e_L L^{-p} (z'_L |ds|^2 + z_L \Delta s - |\bar{K}_L| z_L |ds|) \\ &\geq c' e_L L^{-n} \chi_* \end{aligned} \quad (6.15)$$

when C_* is sufficiently large. Where $z_L = 0$ the factor is constant, so no condition is needed on the original compact part.

This construction supplies the required sign through the joining and bending regions. To control the ADM charge, we continue it into the static tail using its radial flux.

There remains a tail extension of z_L . In the exact static part set $a_L(s) = 1 + M/(2(Ls)^p)$. The radial flux factor is

$$q_L(s) = s^k a_L(s)^2 z_L(s).$$

The equation just used makes $q'_L > 0$ at the beginning of this static part, after increasing C_* if necessary. Extend q_L smoothly so that it stays nondecreasing and is ultimately a positive constant $q_{L,\infty}$. For example, continue its positive derivative over a short interval and then multiply that derivative by a smooth cutoff that becomes zero. This preserves the previous function on an overlap, and all resulting constants remain bounded independently of large L . Define $z_L = q_L/(s^k a_L^2)$ afterwards and $f_L(s) = \int_s^\infty z_L(t) dt$ everywhere. Then $f_L \geq 0$, is constant before the transition, and has a finite value there. The static radial Laplacian satisfies

$$\Delta_{\bar{g}_L} f_L = -L^{-2} a_L^{-2n/p} s^{-k} q'_L(s) \leq 0.$$

It is zero on the ultimate tail, where $\bar{K}_L = 0$.

Take $e_L \rightarrow 0$ sufficiently slowly that $d_L/e_L \rightarrow 0$. Apply the conformal change

$$g_L^* = U_L^{4/p} \bar{g}_L, \quad K_L^* = U_L^{2/p} \bar{K}_L.$$

Equations (6.3), (6.12), and (6.15) prove DEC everywhere for all sufficiently large L . Since U_L is constant on the original core, the future boundary expansion keeps its weak sign. Moreover $U_L \geq 1$, so this repair increases the metric as a quadratic form.

On the static tail, $f_L(s) = q_{L,\infty} p^{-1} s^{-p} + O(L^{-p} s^{-2p})$. It follows that the coefficient of $|y|^{-p}$ in $U_L - 1$ is $e_L q_{L,\infty}/p$. Computed in the asymptotically Euclidean rest coordinates y , the energy is

$$E_L^* = M + \frac{2e_L q_{L,\infty}}{p} = M + o(1), \quad P_L^* = 0. \quad (6.16)$$

One can also see the exact normalization without an expansion: on this tail $a_L U_L$ is a radial Euclidean harmonic function with limit one. Indeed $\Delta_\delta a_L = 0$, and the conformal Laplacian formula gives, for the full factor U_L ,

$$\Delta_\delta(a_L U_L) = a_L^{1+4/p} \Delta_{a_L^{4/p} \delta} U_L + U_L \Delta_\delta a_L = 0.$$

The radial Euclidean harmonic functions tending to one are $1 + c|y|^{-p}$. Thus the ultimate metric is exactly an isotropic Schwarzschild metric with the mass in Equation (6.16). The new constraints are integrable, since the repair source is compactly supported and the ultimate end is vacuum.

6.4 Comparing every enclosing hypersurface

We have obtained zero momentum, energy tending to m , and the dominant energy condition. The numerical transfer also requires a lower bound for the infimum over *all* enclosing hypersurfaces. The metric need not approach the original metric uniformly on the whole end during the change of slice. The following comparison supplies what is needed for all enclosing areas. In the original x coordinates the metrics g_L^* have a uniform lower bound $c_0 \delta$ throughout the region $r \geq L$, where $c_0 > 0$ depends on the fixed boost. Choose a fixed $0 < \lambda < 1$ with $\lambda^2 < c_0/2$.

Let $h_L : [0, \infty) \rightarrow [0, \infty)$ be smooth and increasing, equal to r for $r \leq \sqrt{L}$, with $\lambda \leq h'_L \leq 1$, and with $h'_L = \lambda$ for $r \geq 2\sqrt{L}$. Set $\Phi_L(r\theta) = h_L(r)\theta$ on the end, and extend by the identity on the core. The derivative eigenvalues in the Euclidean radial and tangential directions are h'_L and h_L/r , respectively. They are at most one; for $r \geq L$ they are at most $\lambda + O(L^{-1/2})$, since $h_L(r) = \lambda r + O(\sqrt{L})$. Also $h_L \rightarrow \infty$, so Φ_L is a proper diffeomorphism of the entire exterior onto itself.

For $r \leq \sqrt{L}$ the map is the identity and $g_L^* \geq g$. For $\sqrt{L} \leq r \leq L$, both g at x and g at $\Phi_L(x)$ equal $\delta + o(1)$ uniformly, and g_L^* is a conformal multiple, at least one, of g . The derivative bound therefore gives $\Phi_L^* g \leq (1 + o(1))g_L^*$ there. For $r \geq L$ one instead has

$$\Phi_L^* g \leq (\lambda^2 + o(1))\delta \leq g_L^*.$$

Combining the regions proves

$$\Phi_L^* g \leq (1 + o(1))g_L^*. \quad (6.17)$$

If $\Gamma = \partial D$ is any enclosing hypersurface for g_L^* , then $\Phi_L(D)$ is connected, closed, has interior in $\text{int } \Omega$, and contains the entire distant end. Its full intrinsic boundary is $\Phi_L(\Gamma)$, including every component and every portion on S . Restricting Equation (6.17) to each k -dimensional tangent plane and taking determinants gives

$$\text{Area}_{g_L^*}(\Gamma) \geq (1 + o(1))^{-k/2} \text{Area}_g(\Phi_L(\Gamma)) \geq (1 - o(1))a. \quad (6.18)$$

Taking an infimum uses no existence or regularity of a minimizing cut.

6.5 A strict scalar perturbation

Jaracz used a spacetime Poisson equation to make the dominant energy condition strict by a conformal deformation in three dimensions [29, Theorem 1.1 and Section 3.1, preprint version 1]. The conformal scaling below is its dimension-dependent counterpart. Here a negative inner normal derivative also makes trapping strict; the smoothed drift, prescribed decay, and charge and enclosing-area estimates are established for the present exterior class.

At this stage the inequalities may still be non-strict. We solve a scalar equation with positive source and negative inner normal derivative. The source gives a quantitative strict energy margin; the normal derivative makes the future expansion strictly negative. We state the equation on the original decay class as well as on a static tail, so the resulting scalar perturbation is available independently of the preceding replacement.

Lemma 6.5 (Strict conformal deformation). *Let Ω be a smooth connected exterior, complete with its nonempty compact smooth boundary S included, of dimension $n \geq 3$, and with exactly one Euclidean coordinate end outside a compact set. Use the radius ϱ fixed at the start of this section and the normal ν into Ω . Let (g, K) be smooth up to S , with $g - \delta = O_6(r^{-q})$ and $K = O_5(r^{-1-q})$ for some $q > (n-2)/2$. Choose $0 < \beta < \min(1, q)$ and a smooth nonnegative function $B \geq |K|_g$ with $B = O_5(r^{-1-q})$. Then the problem*

$$-\Delta_g \phi = B \sqrt{|d\phi|_g^2 + \varrho^{-2k} + \varrho^{-n-\beta}}, \quad \partial_\nu \phi = -1 \text{ on } S, \quad \phi \longrightarrow 0 \text{ at infinity} \quad (6.19)$$

has a smooth positive solution, unique among decaying smooth solutions. It satisfies

$$\begin{aligned} \phi &= O_6(r^{2-n}), \quad \frac{-\Delta_g \phi}{r^{-n-\beta}} \longrightarrow 1, \\ \mathfrak{f}_\phi &:= \lim_{R \rightarrow \infty} \int_{S_R} \partial_{\nu_g} \phi \, dA_g \\ &= -\text{Area}_g(S) - \int_\Omega (B \sqrt{|d\phi|^2 + \varrho^{-2k} + \varrho^{-n-\beta}}) \, dV_g < 0. \end{aligned} \quad (6.20)$$

If K is compactly supported, B may be chosen compactly supported. If in addition $g - \delta = O_d(r^{2-n})$ for all d , the solution has this decay through every derivative order.

Suppose also that the ADM limits of (g, K) exist and are finite, that the data satisfy DEC, and that $\theta_+(S) \leq 0$ on every boundary component. For all sufficiently small $\delta > 0$,

$$g_\delta = e^{2\delta\phi} g, \quad K_\delta = e^{\delta\phi} K$$

satisfy

$$\mu_\delta - |J_\delta|_{g_\delta} \geq c_\delta \varrho^{-n-\beta}, \quad \theta_{+,\delta} < 0, \quad E_\delta = E - \frac{\delta}{\omega} \mathfrak{f}_\phi, \quad P_\delta = P. \quad (6.21)$$

Here $g_\delta \geq g$, and their ratio tends uniformly to one as $\delta \downarrow 0$.

Proof. 1. *Continuation on a truncation.* Write $a = \varrho^{-k}$ and $b = \varrho^{-n-\beta}$. The positive regularizing term a^2 makes the right side smooth as a function of $d\phi$, with first gradient derivative of norm at most B . Choose a large coordinate sphere enclosing S and a compact core. On a larger truncation Ω_R impose $\phi = 0$ on S_R , and multiply B in the equation by $t \in [0, 1]$. The inner Neumann and outer Dirichlet boundary components are disjoint smooth hypersurfaces.

At $t = 0$ this mixed Laplace problem is solvable. Indeed its weak formulation is coercive on H^1 functions of zero outer trace by Poincaré's inequality; the Neumann datum is a bounded boundary functional by the trace theorem. Lax–Milgram gives the weak solution and local boundary regularity gives a smooth solution. The linearization at any solution for $0 \leq t \leq 1$ is

$$-\Delta_g u - tB \frac{\langle d\phi, du \rangle_g}{\sqrt{|d\phi|^2 + a^2}}, \quad \partial_\nu u = 0 \text{ on } S, \quad u = 0 \text{ on } S_R.$$

It has a bounded drift, is uniformly elliptic, and has no zeroth order term. Its homogeneous kernel is zero by the strong maximum principle and the Hopf boundary lemma: a nonzero maximum or minimum cannot occur in the interior or on a zero-Neumann component [22, Chapter 3]. The same conclusion follows for weak limits with merely bounded drift. The drift is a compact perturbation of the mixed Laplace isomorphism in Hölder spaces, so its index is zero and this injectivity gives surjectivity. The implicit function theorem gives openness of the continuation set.

2. Bounds independent of the truncation. We detail the estimates giving closedness and exhaustion. All solutions are nonnegative. An interior negative minimum contradicts $-\Delta\phi > 0$. At a minimum on S the Hopf lemma requires a negative outward derivative, whereas the outward derivative of Ω_R is $-\partial_\nu\phi = 1$. The outer boundary value is zero.

For $0 < \alpha < \min(\beta, q)$ and sufficiently large r_0 , the function

$$W(r) = r^{-p}(1 - r^{-\alpha}), \quad r \geq r_0,$$

is positive and satisfies

$$-\Delta_g W - B|dW|_g \geq cr^{-n-\alpha}. \quad (6.22)$$

In fact $-\Delta_\delta W = \alpha(p + \alpha)r^{-n-\alpha}$, the metric error is $O(r^{-n-q})$, and $B|dW| = O(r^{-n-q})$. Since $b + Ba = O(r^{-n-\beta})$, a sufficiently large multiple $C(A+1)W$, where $A = \sup_{S_{r_0}} \phi$, is a supersolution outside r_0 , dominates ϕ on S_{r_0} , and is nonnegative on S_R . Here we used $\sqrt{|d\phi|^2 + a^2} \leq |d\phi| + a$ and the fact that this gradient dependence is Lipschitz. Subtracting a supersolution therefore gives a scalar linear inequality with bounded drift, to which comparison applies. Hence

$$0 \leq \phi(x) \leq C(1 + A)r^{-p} \quad (r_0 \leq r \leq R), \quad (6.23)$$

uniformly in R and t .

We claim that the suprema of all these solutions are uniformly bounded. Otherwise divide a sequence by its suprema $M_i \rightarrow \infty$. Estimate (6.23) ensures that the normalized functions have a positive supremum on a fixed compact region. Local W^{2,a_0} estimates, for any fixed $a_0 > n$, apply uniformly, including the inner Neumann boundary: the equation has right side bounded by a fixed multiple of $|d\phi| + 1$, and first derivative interpolation absorbs the gradient term in the second derivative estimate. The same argument applies after division by M_i . Compactness gives a nonnegative, nonzero limit u with $u \rightarrow 0$ at infinity and

$$-\Delta_g u = t_* B|du|, \quad \partial_\nu u = 0.$$

If the outer radii instead have a finite limit, the limit has zero data at that limiting sphere. In either case its positive maximum is attained. The strong maximum principle, or the Hopf lemma at S , contradicts that maximum. These principles apply because $B|du|$ equals a bounded measurable drift paired with du , defining that drift to be zero where $du = 0$. This proves the uniform supremum estimate.

3. Exhaustion and uniqueness. The W^{2,a_0} estimates now give uniform $C^{1,\gamma}$ bounds on fixed patches. Their right sides are then Hölder continuous, and the scalar Schauder estimates give $C^{2,\gamma}$ bounds. Differentiation and the strictly positive regularizer a give all higher local bounds. For each fixed truncation these estimates close continuation to $t = 1$. Letting $R \rightarrow \infty$ gives a smooth solution of Equation (6.19); the tail bound gives the specified limit. It is strictly positive by the same minimum principle. The difference of two decaying solutions has zero Neumann data and solves a homogeneous equation with bounded drift, by integrating the gradient derivative of

the square root along the segment between their gradients. Its positive and negative extrema are excluded as above, proving uniqueness.

4. *End decay and flux.* On dyadic end annuli rescale $x = Lz$ and $\Phi(z) = L^p\phi(Lz)$. Estimate (6.23) bounds Φ . The rescaled equation has uniformly controlled metric coefficients and the form

$$-\Delta_{g(Lz)}\Phi = O_5(L^{-q})\sqrt{|d\Phi|_{g(Lz)}^2 + |z|^{-2k} + L^{-\beta}|z|^{-n-\beta}},$$

where the coefficient denoted $O_5(L^{-q})$ is nonnegative and smooth. The same interpolation, scalar estimates and differentiation give uniform bounds for six derivatives of Φ . In the compactly supported B case the equation is just a Poisson equation on the distant end, and all available metric derivative orders give corresponding derivative bounds. This proves the asserted decay. It also gives $B\sqrt{|d\phi|^2 + r^{-2k}} = O(r^{-n-q})$, so $\beta < q$ proves the Laplacian ratio in Equation (6.20).

The source is integrable. Integrating the equation on Ω_R , remembering that its inner outward normal is $-\nu$, gives

$$\int_{S_R} \partial_{\nu_g} \phi dA_g = -\text{Area}_g(S) - \int_{\Omega_R} (B\sqrt{|d\phi|^2 + a^2 + b}) dV_g.$$

This proves the claimed finite negative flux.

5. *Strict inequalities and charges.* Finally $|d\phi|^2 \leq Cb$ globally: on the end the quotient has order $r^{-(n-2-\beta)}$, which is bounded because $\beta < 1 \leq n-2$, and on the core b is positive. Equation (6.19) implies, for every unit-ball vector v , $-\Delta\phi + K(\nabla\phi, v) \geq b$. Use Equation (6.2) with $s = \delta\phi$; for sufficiently small fixed $\delta > 0$ the negative square term is at most half the positive $k\delta b$ term. This proves the strict DEC margin, and $e^{\delta\phi}\theta_{+,\delta} = \theta_+ - k\delta < 0$. The conformal energy vector changes by $-2k\delta d\phi + o(r^{1-n})$. The geometric and Euclidean fluxes of $d\phi$ agree in the limit, since their difference is $O(r^{-q})$ after integration. Division by $2k\omega$ gives the energy formula in Equation (6.21) exactly. The momentum trace reversal is multiplied by $e^{\delta\phi}$; its flux change is $O(r^{-q}) \rightarrow 0$. Positivity and boundedness of ϕ give the last metric statements. $\square$

Proof of Proposition 6.1. Apply Lemma 6.5 to the repaired data (g_L^*, K_L^*) , choosing B compactly supported and choosing $\delta_L \downarrow 0$ so that $\delta_L \|\phi_L\|_\infty \rightarrow 0$ and $\delta_L |\mathfrak{f}_{\phi_L}| \rightarrow 0$. There is no need for constants in this solve to be uniform in L : this choice is made after the entire replacement at that L . Its scalar curvature has the required decay, since the ultimate initial end is scalar flat and

$$R_{e^{2\delta_L\phi_L}g_L^*} = e^{-2\delta_L\phi_L}(2k\delta_L r^{-n-\beta} - kp\delta_L^2 |d\phi_L|^2)$$

there. Compact support of K_L^* is preserved, and all metric derivative orders have the stated asymptotics. The strict margin and expansion are those of Lemma 6.5. The core remains smooth and the tail is uniformly Euclidean comparable, so completeness with S included is preserved. The constraints are integrable on the core and have integrable decay on the tail.

By Equation (6.16) and the choice of δ_L , the final energy tends to $M = m > 0$, and the final momentum is zero. The strict conformal deformation only increases the metric, so Equation (6.17) and the enclosing-area comparison remain valid; in particular, every resulting enclosing-area infimum is positive. Pass to a sequence along which the comparison errors are bounded by numbers $\epsilon_j \downarrow 0$. These data satisfy Definition 2.1 and every assertion in Equation (6.1). $\square$

6.6 Increasing energy while fixing momentum

The reduction above uses a fixed original future-timelike ADM vector. The following independent conformal change applies without a timelike assumption: it raises the energy by a prescribed amount, fixes the momentum, and increases the area of every enclosing hypersurface. These are the properties used in the proof of Corollary 1.4 in Section 2.

Lemma 6.6 (Increasing energy under the strong decay assumptions). *Assume all hypotheses of Theorem 1.3 except possibly $E > |P|_\delta$. For every $\lambda > 0$ there is a smooth function $U_\lambda \geq 1$, constant near S , such that*

$$g_\lambda = U_\lambda^{4/p} g, \quad K_\lambda = U_\lambda^{2/p} K$$

are smooth complete one-ended data with the same O_6/O_5 decay and q , integrable constraint densities, the dominant energy condition and weakly future outer trapped boundary, and satisfy

$$E_\lambda = E + 2\lambda, \quad P_\lambda = P, \quad A_{\min, g_\lambda}(S) \geq A_{\min, g}(S) > 0.$$

Proof. For $U > 0$ put $a = U^{2/p}$ and $(g_U, K_U) = (a^2 g, aK)$. Substitution of $s = 2p^{-1} \log U$ into Equation (6.2) gives, as scalar and covector identities,

$$a^2 \mu_U = \mu - \frac{2k}{pU} \Delta_g U, \quad a J_U = J + \frac{2k}{pU} K(\nabla U, \cdot).$$

Indeed, the scalar terms combine as $-k\Delta s - \frac{kp}{2}|ds|^2 = -\frac{2k}{pU} \Delta U$. Taking the new-metric covector norm introduces another factor a^{-1} , so

$$a^2(\mu_U - |J_U|_{g_U}) \geq \mu - |J|_g + \frac{2k}{pU}(-\Delta_g U - |K|_g |dU|_g). \quad (6.24)$$

The boundary identity is

$$a\theta_{+,U} = \theta_+ + \frac{2k}{pU} \partial_\nu U.$$

Thus a factor constant near S preserves its expansion sign, and $-\Delta_g U \geq |K|_g |dU|_g$ preserves the dominant energy condition.

Choose $0 < \xi < \min(q, 1)$. In a sufficiently distant part of the coordinate end set

$$T(r) = r^{-p} - r^{-p-\xi}.$$

There $T > 0$ and $T' < 0$. Since $p = n - 2$, $\Delta_\delta r^{-\alpha} = \alpha(\alpha + 2 - n)r^{-\alpha-2}$. The original metric and tensor estimates therefore give

$$-\Delta_g T - |K|_g |dT|_g = \xi(p + \xi)r^{-n-\xi} + O(r^{-n-q}) > 0$$

after increasing the radius. Choose a nondecreasing smooth switch β from zero to one in this region, constant near its endpoints, and put

$$G(r) = - \int_r^\infty \beta(t) T'(t) dt.$$

Extend G constantly through the interior. Then $G \geq 0$, it is constant near S , and $G = T$ sufficiently far out. Because $dG = \beta dT$,

$$-\Delta_g G - |K|_g |dG|_g = \beta(-\Delta_g T - |K|_g |dT|_g) - \beta' T' |dr|_g^2 \geq 0.$$

Use $U_\lambda = 1 + \lambda G$. Equation (6.24) and the boundary formula prove the required signs for every fixed $\lambda > 0$, without a smallness condition.

The factor is bounded for each fixed λ and is at least one, so completeness with S included is preserved. The manifold and its unique coordinate end are unchanged. On the tail $G = O_j(r^{-p})$ at every fixed derivative order. Since $q < p$, the added metric tail is faster than the original decay, and

$$g_\lambda - \delta = O_6(r^{-q}), \quad K_\lambda = O_5(r^{-1-q}).$$

Moreover,

$$\Delta_g U_\lambda = O(r^{-n-\xi}) + O(r^{-n-q}), \quad |K|_g |dU_\lambda|_g = O(r^{-n-q}).$$

These terms are integrable against $dV_g \asymp r^{n-1} dr d\omega$. The scalar and covector identities above, and the bounded conformal factors for fixed λ , show that the new constraint densities are integrable against dV_{g_λ} .

For the charges write $f_\lambda = U_\lambda^{4/p}$. Then

$$f_\lambda = 1 + \frac{4\lambda}{p} G + O_1(r^{-2p}).$$

In the metric flux of Equation (1.2), the leading change is $-k\partial_i(f_\lambda - 1)$. Terms involving $g - \delta$ or its first derivatives have integrated flux $O(R^{-q})$, and the nonlinear remainder has flux $O(R^{-p})$. Since

$$\lim_{R \rightarrow \infty} \int_{S_R} \partial_r G dA_\delta = -p\omega_{n-1},$$

the current normalization gives

$$E_\lambda - E = -\frac{2\lambda}{p\omega_{n-1}} \lim_{R \rightarrow \infty} \int_{S_R} \partial_r G dA_\delta = 2\lambda.$$

If $\pi = K - (\text{tr}_g K)g$, its transformed tensor is exactly $\pi_\lambda = U_\lambda^{2/p} \pi$. The additional momentum flux in Equation (1.3) is $O(R^{-q})$, because $U_\lambda^{2/p} - 1 = O(r^{-p})$ and $\pi = O(r^{-1-q})$. Thus $P_\lambda = P$, and the new ADM limits are finite.

Finally, the admissible domains in Definition 1.2 are unchanged. For their entire intrinsic boundaries,

$$\text{Area}_{g_\lambda}(\partial D) = \int_{\partial D} U_\lambda^{2k/p} dA_g \geq \text{Area}_g(\partial D).$$

This comparison includes every component and boundary-coincident portion. Taking the infimum proves the claimed enclosing-area inequality. $\square$

6.7 An infinitesimal consequence

The function ϕ from Lemma 6.5 also gives a strict first variation at points where the dominant energy condition is saturated. Write $\mathfrak{E}_\delta(v) = 2(\mu_\delta + J_\delta(e^{-\delta\phi}v))$. For a vector v with $|v|_g \leq 1$ and $\mu + J(v) = 0$, differentiation of Equation (6.2) gives

$$\mathfrak{E}'_0(v) = 2k(-\Delta\phi + K(\nabla\phi, v)) \geq 2k\varrho^{-n-\beta}. \quad (6.25)$$

Its quotient by $r^{-n-\beta}$ tends to $2k$ along any sequence of such vectors based at points escaping to infinity: the two terms involving B and K are $O(r^{-n-q})$. At a marginal boundary the expansion derivative of the same direction is exactly $-k$.

7 Two metric derivatives and one tensor derivative in dimension three

Theorem 2.3 can be applied under only two differentiated metric bounds and one differentiated tensor bound in dimension three. The reason is a separate end construction: first match the original ADM vector by a Schwarzschild reference slice, then bend that slice to rest and repair the small constraint error. The metric changes at distances tending to infinity. We prove compactness of minimizing enclosures before making those changes; it gives convergence of the enclosing-area infimum, not only a one-sided area bound. A positive conformal family subsequently removes timelikeness as an assumption, and truncation of the other ends gives the complete-data statement.

7.1 Initial data, normals, and enclosing area

We work in three spatial dimensions, with zero cosmological constant and $G = c = 1$. In this section μ, J denote the physical densities: the geometric densities of Equation (1.1) are $8\pi\mu, 8\pi J$. The ADM energy and momentum keep their previous normalization. Thus, for the metric g and second fundamental form K ,

$$16\pi\mu = R_g + (\operatorname{tr}_g K)^2 - |K|_g^2, \quad 8\pi J = \operatorname{div}_g(K - (\operatorname{tr}_g K)g). \quad (7.1)$$

Here J is a covector field. The dominant energy condition is

$$\mu \geq |J|_g. \quad (7.2)$$

All initial data considered here are smooth, including at a compact boundary when one is present.

On an asymptotically flat end with Euclidean coordinates x and $r = |x|$, our basic assumptions are

$$g_{ij} - \delta_{ij} = O_2(r^{-q}), \quad K_{ij} = O_1(r^{-1-q}), \quad q > \frac{1}{2}. \quad (7.3)$$

The notation $O_j(r^{-a})$ includes the corresponding coordinate derivative bounds through order j . We assume that μ and $|J|_g$ are integrable and that the following ADM limits exist and are finite:

$$E = \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{S_r} (\partial_j g_{ij} - \partial_i g_{jj}) n^i dA_\delta, \quad (7.4)$$

$$P_i = \frac{1}{8\pi} \lim_{r \rightarrow \infty} \int_{S_r} (K_{ij} - (\operatorname{tr}_g K)g_{ij}) n^j dA_\delta. \quad (7.5)$$

The normal and area form in these definitions are Euclidean. Whenever $E > |P|_\delta$, write

$$m = \sqrt{E^2 - |P|_\delta^2}. \quad (7.6)$$

For a two-sided surface Σ with specified unit normal ν , we use

$$H_\Sigma = \operatorname{div}_\Sigma \nu, \quad \theta_+(\Sigma) = H_\Sigma + \operatorname{tr}_\Sigma K.$$

Thus Euclidean spheres have positive mean curvature for the normal toward infinity. A future marginally outer trapped surface, abbreviated MOTS, satisfies $\theta_+ = 0$. A weakly future outer trapped surface satisfies $\theta_+ \leq 0$. Our spacetime sign convention is

$$K(Y, Z) = \mathbf{g}(\nabla_Y n, Z), \quad (7.7)$$

where n is the future unit timelike normal. Equivalently, in Gaussian normal coordinates the spatial metric has normal derivative $2K$.

Definition 7.1 (Exterior and enclosing area). An exterior is a connected orientable smooth three-manifold Ω with nonempty compact smooth boundary, complete as a metric space with its boundary included, and with exactly one end, which is asymptotically flat. An enclosing cut is the entire compact smooth embedded two-sided intrinsic boundary of a connected closed outer domain containing the sufficiently distant end, with its manifold interior in the open exterior. Equivalently, it separates the entire inner boundary from infinity, with bounded complementary pockets filled on the inner side. Every frontier component is counted, including portions coinciding with the inner boundary. A cut may have several components. Its normal points toward the retained end. We put

$$a_g(\partial\Omega) = \inf\{|\Gamma|_g : \Gamma \text{ is an enclosing cut}\}.$$

Boundary coincidence is needed because a minimum and its first variations may involve contact with the inner boundary. Disconnected cuts are needed because minimizing hulls and intermediate trapped boundaries may be disconnected. Lemma 7.8 proves that minimizing perimeter among filled inner sets, counting contact area, gives the same infimum and supplies the compactness needed below.

7.2 The one-ended exterior inequality

Theorem 7.2 (Minimum-enclosing-area inequality). *Let (Ω, g, K) be an exterior as in Definition 7.1. Assume Equations (7.1)–(7.3), integrability of μ and $|J|_g$, and the finite ADM limits (7.4)–(7.5). Orient $\partial\Omega$ by the normal into Ω , and suppose $\theta_+(\partial\Omega) \leq 0$. If $E > |P|_\delta$, then*

$$\sqrt{E^2 - |P|_\delta^2} \geq \sqrt{\frac{a_g(\partial\Omega)}{16\pi}}. \quad (7.8)$$

No extension of the data across $\partial\Omega$ is required.

7.3 A positive lower bound for every enclosing cut

Before preparing the end, we record a geometric fact about the full enclosing infimum. A thin tube joining the inner boundary to infinity carries a fixed flux through every enclosing cut. A bound for the flux density then gives a uniform positive area bound. This argument uses neither the constraint equations nor a sign for the boundary expansion.

Lemma 7.3 (Positive enclosing area). *Let (Ω, g) be a connected orientable smooth three-dimensional exterior, complete with its nonempty compact smooth boundary B included, with one asymptotically flat end and compact complement. Let $a_g(B)$ be the infimum of the full areas of the enclosing cuts in Definition 7.1, including all components and all portions coincident with B . Then*

$$0 < a_g(B) < \infty.$$

Proof. A sufficiently large coordinate sphere encloses B , so the infimum is finite. To obtain a lower bound, choose a point of B and a smooth proper embedded ray starting there, initially normal to B and pointing into Ω . Choose the ray to be straight in the asymptotic coordinates outside a compact set. A tubular neighborhood has product coordinates $(t, y) \in [0, \infty) \times D^2$, with its initial disk contained in B . On the straight tail the transverse coordinate disk can have a fixed positive width.

Choose a smooth two-form η supported in the interior of D^2 with $\int_{D^2} \eta = 1$. Pull it back by $(t, y) \mapsto y$ and extend it by zero across the lateral boundary of the tube. This gives a smooth closed two-form ω on Ω , smooth also at B . Its pointwise comass is bounded: this follows from compactness on the initial portion of the tube and from asymptotic flatness and fixed coordinate width on the tail. Write

$$C_\omega = \sup_{x \in \Omega} \sup_{v, w \text{ orthonormal in } (T_x \Omega, g)} |\omega_x(v, w)| < \infty.$$

Orient the transverse disk so that the flux through a sufficiently large outward-oriented coordinate sphere S_R is one.

Let Γ be a smooth enclosing cut lying in the interior of Ω , oriented toward the end, and choose S_R outside it. The region between Γ and S_R is compact and disjoint from B ; all of its inner boundary components are counted in Γ . Stokes' theorem gives

$$\int_\Gamma \omega = \int_{S_R} \omega = 1, \quad 1 \leq C_\omega \text{Area}_g(\Gamma).$$

This calculation also applies when the cut is disconnected. If a smooth cut touches or coincides with B , push it into the interior by a smooth collar flow pointing from B into Ω , filling the collar on the obstacle side. The resulting surfaces are enclosing cuts and their areas converge to the full area of the original cut. Thus the same lower bound holds for boundary-coincident cuts. Taking the infimum yields $a_g(B) \geq C_\omega^{-1} > 0$. The equivalence proved in Lemma 7.8 transfers this bound to the filled-perimeter formulation as well. $\square$

For the end construction, retain the exterior and boundary orientation of Theorem 7.2 and write $B = \partial\Omega$. In particular, the single Euclidean coordinate end has compact complement. We will use the decay bounds

$$g - \delta = O_2(r^{-q}), \quad K = O_1(r^{-1-q}), \quad q > \frac{1}{2}. \quad (7.9)$$

The density, finite-charge and weak-trapping hypotheses are those stated in Theorem 7.2. The preparation assumes $E > |P|$; the conformal construction below does not.

7.4 Preparation statements and the one-ended deductions

The end construction will supply the following precise input to Theorem 2.3. Its proof is completed in Section 7.12.

Proposition 7.4 (Reduction to strict data with a controlled rest end). *Let (Ω, g, K) satisfy the exterior hypotheses of Theorem 7.2, including $\theta_+ \leq 0$ on its smooth compact boundary, and let $m = (E^2 - |P|^2)^{1/2} > 0$. There is a sequence of smooth exterior data (g_j, K_j) on Ω with the following properties:*

- (i) $\mu_j > |J_j|_{g_j}$ everywhere and $\theta_{+,j} < 0$ on B . The constraint densities are integrable. On the end their DEC margin is bounded below by $c_j r^{-3-\delta}$ for some $c_j > 0$ and one fixed $0 < \delta < 1$. For a smooth positive radius extension ρ , the constant can be decreased so that $\mu_j - |J_j|_{g_j} \geq c_j \rho^{-3-\delta}$ on all of Ω .
- (ii) K_j is compactly supported, and in a rest chart $g_j - \delta = O_k(r^{-1})$ and $R_{g_j} = O_k(r^{-3-\delta})$ for every k . In particular $\text{Ric}_{g_j} = O(r^{-3})$ there. Their ADM momentum is zero.

- (iii) $(g_j, K_j) \rightarrow (g, K)$ smoothly on every fixed compact subset up to the boundary, their ADM energies satisfy $E_j \rightarrow m$, and $a_{g_j}(B) \rightarrow a_g(B)$.

The next lemma uses the original exterior, decay, integrability and finite-charge hypotheses of Theorem 7.2, with no sign restriction on the ADM vector. It raises energy with fixed momentum and controls the area for every finite value of its parameter. It will remove the timelike assumption after the timelike inequality has been proved. Its proof is in Section 7.7.

Lemma 7.5 (Strict direction under the original decay). *Assume the exterior, decay, integrability and finite-charge hypotheses specified above, with $\mu \geq |J|_g$. Fix $0 < \delta < \min(q, 1)$. There are smooth positive functions w and ϕ on Ω , with $w = r^{-3-\delta}$ on the end, such that*

$$\partial_\nu \phi = -1 \quad \text{on } B, \quad \phi = O_2(r^{-1}), \quad (7.10)$$

$$-\Delta_g \phi - |K|_g |d\phi|_g \geq w, \quad \frac{-\Delta_g \phi}{w} \rightarrow 1. \quad (7.11)$$

The Euclidean normal flux

$$L_\phi = \lim_{r \rightarrow \infty} \int_{S_r} \partial_r \phi \, dA_\delta$$

is finite and negative. For every finite $s \geq 0$, set

$$u_s = 1 + s\phi, \quad g_s = u_s^4 g, \quad K_s = u_s^2 K. \quad (7.12)$$

These data are complete up to B and satisfy DEC, strictly for $s > 0$. If $\theta_+ \leq 0$ on B , their boundary expansion remains nonpositive and is strictly negative for $s > 0$. Their decay exponent can be taken to be $\min(q, 1) > 1/2$, their constraint densities are integrable, and

$$E_s = E - \frac{sL_\phi}{2\pi}, \quad P_s = P. \quad (7.13)$$

Their enclosing-area infima satisfy

$$a_g(B) \leq a_{g_s}(B) \leq (1 + s\|\phi\|_\infty)^4 a_g(B). \quad (7.14)$$

In particular $E_s \rightarrow E$ and $a_{g_s}(B) \rightarrow a_g(B)$ as $s \downarrow 0$. No sign assumption on the ADM vector is needed.

If K is compactly supported and the metric has symbol estimates of all orders with $g - \delta = O(r^{-1})$, the function can instead be chosen with $\phi = O_k(r^{-1})$ for every k , while retaining Equations (7.10) and (7.11).

7.5 Applying the energy inequality at each fixed stage

Proof of Theorem 7.2. Let $m = \sqrt{E^2 - |P|^2} > 0$, and take the sequence in Proposition 7.4. Each datum is a smooth complete one-ended exterior on Ω with the same nonempty compact boundary B . The proposition gives a compactly supported tensor, $O_d(r^{-1})$ metric decay for every fixed derivative order d , scalar curvature $O(r^{-3-\delta})$ with $0 < \delta < 1$, integrable constraints, strict future outer trapping, and a global weighted DEC margin. These are the geometric and asymptotic conditions in Definition 2.1.

The enclosing hypersurfaces in Definition 7.1 are also those in Definition 1.2: both definitions retain a connected end-containing outer domain and count its entire intrinsic frontier, including

coincidence with the obstacle. Filling pockets removes only surplus frontier and leaves either infimum unchanged. Thus the area for the input metric is $A_j = a_{g_j}(B) > 0$, where positivity follows from Lemma 7.3.

There is one normalization conversion. The densities used in this section are physical densities, while those of Theorem 2.3 satisfy $2\mu_A = R + (\text{tr } K)^2 - |K|^2$ and $J_A = \text{div}(K - (\text{tr } K)g)$. Hence

$$\mu_A = 8\pi\mu, \quad J_A = 8\pi J.$$

The global weighted margin becomes $\mu_{A,j} - |J_{A,j}|_{g_j} \geq 8\pi c_j \rho^{-3-\delta}$, and all sign and integrability conditions are preserved. At $n = 3$, $k = n - 1 = 2$ and $\omega_2 = 4\pi$, so the energy and momentum prefactors in that theorem are respectively

$$\frac{1}{2k\omega_2} = \frac{1}{16\pi}, \quad \frac{1}{k\omega_2} = \frac{1}{8\pi}.$$

Thus neither ADM energy nor momentum nor invariant mass is rescaled.

Theorem 2.3 therefore applies to each fixed input datum and gives

$$E_j \geq \frac{1}{2} \left(\frac{a_{g_j}(B)}{4\pi} \right)^{1/2} = \sqrt{\frac{a_{g_j}(B)}{16\pi}}.$$

Its elliptic exhaustion and its numerical limits in N and ϵ have already been completed for that datum; no estimate uniform in j is required. Now use $E_j \rightarrow m$ and the full convergence $a_{g_j}(B) \rightarrow a_g(B)$ to obtain

$$\sqrt{E^2 - |P|^2} \geq \sqrt{\frac{a_g(B)}{16\pi}},$$

as asserted. □

7.6 Future timelikeness from the exterior inequality

The conformal direction in Lemma 7.5 allows us to move the ADM energy upward while fixing the momentum and increasing every enclosing area. If the original ADM vector were not future timelike, we could approach the future null cone through data to which the timelike inequality applies. The positive area bound would persist along this family, giving a contradiction.

Theorem 7.6 (The exterior inequality without a timelike assumption). *Let (Ω, g, K) satisfy all the hypotheses of Theorem 7.2 except $E > |P|_\delta$. Then*

$$E > |P|_\delta, \quad \sqrt{E^2 - |P|_\delta^2} \geq \sqrt{\frac{a_g(B)}{16\pi}}, \quad B = \partial\Omega.$$

No extension of the data across B is required.

Proof. Lemma 7.5 supplies a positive function ϕ with negative flux L_ϕ , and, for every finite $s \geq 0$, the data

$$g_s = (1 + s\phi)^4 g, \quad K_s = (1 + s\phi)^2 K.$$

These data retain completeness, integrable constraint densities, DEC, weak future outer trapping of B , and the required differentiated decay. The lemma makes no sign assumption on the original ADM vector. Its charge and area identities, Equations (7.13) and (7.14), give

$$E_s = E - sL_\phi/(2\pi), \quad P_s = P, \quad a_{g_s}(B) \geq a_g(B) > 0,$$

where the last strict inequality is Lemma 7.3.

Suppose $E \leq |P|_\delta$, and set $s_0 = 2\pi(|P|_\delta - E)/(-L_\phi)$. This is a finite nonnegative number. For every $s > s_0$ the deformed vector is future timelike, so Theorem 7.2, applied to these fixed data, implies

$$\sqrt{(E - sL_\phi/(2\pi))^2 - |P|_\delta^2} \geq \sqrt{\frac{a_{g_s}(B)}{16\pi}} \geq \sqrt{\frac{a_g(B)}{16\pi}} > 0.$$

As $s \downarrow s_0$ the left side tends to zero, while the final lower bound is fixed and positive. This contradiction proves $E > |P|_\delta$. The timelike exterior inequality now applies directly to (Ω, g, K) and gives the stated bound. $\square$

7.7 A strict conformal direction

Jaracz developed conformal deformations to the strict dominant energy condition using a spacetime Poisson equation [29]. Here a regularized drift equation with nonzero inner Neumann data also gives strict future outer trapping, an exact energy slope, unchanged momentum, and comparison of all enclosing cuts. These quantitative conclusions are established below.

Lemma 7.7 (Conformal constraint formulas). *Let g be a smooth three-dimensional Riemannian metric, K a smooth symmetric covariant two-tensor, and f a smooth real function. Use the physical density normalization in Equation (7.1), and set*

$$g_f = e^{4f}g, \quad K_f = e^{2f}K.$$

The corresponding constraint densities satisfy

$$16\pi e^{4f}\mu_f = 16\pi\mu - 8\Delta_g f - 8|df|_g^2, \quad (7.15)$$

$$8\pi J_f = e^{-2f}(8\pi J + 4K(\nabla f, \cdot)). \quad (7.16)$$

Here the second identity is an identity of covectors. Consequently,

$$\begin{aligned} 16\pi e^{4f}(\mu_f - |J_f|_{g_f}) &\geq 16\pi(\mu - |J|_g) \\ &\quad + 8(-\Delta_g f - |df|_g^2 - |K|_g|df|_g). \end{aligned} \quad (7.17)$$

On the fixed boundary,

$$\theta_{+,f} = e^{-2f}(\theta_+ + 4\partial_\nu f). \quad (7.18)$$

Equivalently, if $u = e^f > 0$, the last summand in Equation (7.17) is $8u^{-1}(-\Delta_g u - |K|_g|du|_g)$.

Proof. Apply the pointwise identities in Equation (6.2) with $n = 3$, $k = 2$, $p = 1$, and $s = 2f$. The densities in that identity are $\mu_A = 8\pi\mu$ and $J_A = 8\pi J$ in this section's physical normalization. Setting the vector argument to zero gives $8\pi e^{4f}\mu_f = 8\pi\mu - 4\Delta_g f - 4|df|_g^2$; multiplication by two is Equation (7.15). The covector formula proved there gives $8\pi J_f = e^{-2f}(8\pi J + 4K(\nabla f, \cdot))$. Its new-metric norm introduces a further factor e^{-2f} , so the triangle inequality gives Equation (7.17) with precisely its factor 8. The boundary formula has $k\partial_\nu s = 4\partial_\nu f$ and proves Equation (7.18). Finally Equation (6.3) uses $U = e^f = u$ here and gives the stated positive-factor form. These are pointwise identities; no decay or additional differentiated end hypothesis is used. $\square$

Proof of Lemma 7.5. Choose β with $\delta < \beta < \min(q, 1)$. Choose smooth $b \geq |K|_g$, positive everywhere and equal to $b_0 r^{-1-\beta}$ sufficiently far out, where $b_0 > 0$ is chosen large enough. This is possible because $\beta < q$. Choose a smooth positive regularizer ζ , equal to r^{-2} on the end, and extend w smoothly and positively to Ω . We solve

$$-\Delta_g \phi = b \sqrt{|d\phi|_g^2 + \zeta^2} + w, \quad \partial_\nu \phi = -1, \quad \phi \longrightarrow 0. \quad (7.19)$$

The regularizer makes this equation smooth even at critical points.

First truncate at a large coordinate sphere S_R and prescribe $\phi = 0$ there. The Neumann and Dirichlet conditions occur on disjoint boundary components. For $0 \leq t \leq 1$, replace b in Equation (7.19) by tb . At $t = 0$, the mixed Laplacian is invertible: its Dirichlet face gives the Poincare inequality, and the weak solution obtained from its coercive quadratic form is smooth by boundary regularity. At any solution, the linearization is a Laplacian with a smooth drift of norm at most b and with the homogeneous mixed boundary conditions. The strong maximum principle and the Hopf boundary principle make its kernel zero. It is a Fredholm perturbation of the mixed Laplacian of index zero, and is therefore invertible. The boundary estimates used here are the ordinary elliptic estimates for Dirichlet and normal Neumann conditions: positive definiteness of g verifies their principal complementing condition, and the disjoint boundary components can be covered separately [1]. The maximum principles and local elliptic estimates below are used in their usual uniformly elliptic forms [22].

To close continuation at $t = 1$, first work on a fixed truncation. The elliptic $W^{2,p}$ estimate and gradient interpolation give

$$\|\phi\|_{W^{2,p}} \leq C_p(1 + \|\phi\|_{L^p} + \|d\phi\|_{L^p}) \leq \frac{1}{2}\|\phi\|_{W^{2,p}} + C'_p(1 + \|\phi\|_{L^p}).$$

The constants include the fixed boundary data and are uniform in t . If the supremum norms of solutions on this truncation were unbounded, divide them by their supremum norms. The preceding estimate with $p > 3$, compactness, and the equation give a nonzero limit v with homogeneous mixed data satisfying

$$-\Delta_g v = t_* b |dv|_g.$$

This can be written $\Delta_g v + A \cdot dv = 0$ with a bounded measurable drift: take $A = t_* b \nabla v / |dv|$ off the critical set and zero on it. The strong and Hopf maximum principles exclude a nonzero solution with the homogeneous mixed conditions. Thus the supremum norms are bounded. The $W^{2,p}$ estimate, Schauder estimates, and the smooth equation now give the bounds needed for closedness of continuation. Existence on each truncation follows.

Every such solution is nonnegative. A negative minimum cannot be interior because the right side of Equation (7.19) is positive. At an inner-boundary minimum, the outward-domain derivative would be negative by the Hopf principle, whereas the prescribed derivative in that direction is 1. The outer value is zero.

It remains to make these bounds independent of R . For fixed sufficiently large A and then sufficiently large r_0 , the positive radial function

$$v_0(r) = r^{-1}(1 - Ar^{-\delta})$$

satisfies, on $r \geq r_0$,

$$-\Delta_g v_0 - b |dv_0|_g = A\delta(1 + \delta)r^{-3-\delta} + O(r^{-3-q}) + O(r^{-3-\beta}) \geq c_0 w.$$

The constants are fixed independently of the truncation. Since $b\zeta = o(w)$, a sufficiently large multiple of v_0 is a supersolution of the regularized equation. The comparison principle gives

$$0 \leq \phi(x) \leq C(1 + M_R)v_0(r), \quad M_R = \sup_{\Omega \cap \{r \leq r_0\}} \phi, \quad r_0 \leq r \leq R. \quad (7.20)$$

The notation for the compact core includes all points outside the end chart. Were M_R unbounded along expanding truncations, the normalized solutions ϕ/M_R would have locally uniform $W^{2,p}$ bounds, including at B , by the preceding local estimates and Equation (7.20). A subsequence would converge locally in C^1 to a nonnegative function with maximum one on the core, homogeneous inner Neumann data, and a bounded-drift homogeneous equation. Equation (7.20) makes its value tend to zero at infinity. It consequently attains a positive maximum, contradicting the strong or Hopf principle for that homogeneous equation. This proves the uniform core bound. The contradiction uses the homogeneous bounded-drift equation.

Local compactness and diagonal extraction give a smooth solution of Equation (7.19), with $\phi = O(r^{-1})$. The claimed derivative count can be checked directly on annuli. Put

$$\phi_\rho(y) = \rho\phi(\rho y), \quad g_\rho(y) = g(\rho y), \quad 1 < |y| < 4.$$

On smaller fixed annuli the equation is

$$-\Delta_{g_\rho}\phi_\rho = \rho b(\rho y) \sqrt{|d\phi_\rho|_{g_\rho}^2 + (\rho^2\zeta(\rho y))^2} + \rho^3 w(\rho y).$$

Its coefficients have uniform $C^{1,1}$ bounds from the two derivatives in Equation (7.9); the scaled drift is $O(\rho^{-\beta})$ and the scaled source is $O(\rho^{-\delta})$. The bounded zeroth-order norm, interior $W^{2,p}$ estimates, and gradient interpolation first give uniform $C^{1,\alpha}$ bounds for some $\alpha > 0$. The right side is then uniformly C^α . Schauder estimates give uniform $C^{2,\alpha}$ bounds on smaller annuli. This proves $\phi = O_2(r^{-1})$ without using a third derivative of g or a second derivative of K .

The right side of Equation (7.19) is $w + O(r^{-3-\beta})$. It is integrable and proves Equation (7.11). The divergence theorem, with outward-domain normal $-\nu$ on B , gives

$$\lim_{r \rightarrow \infty} \int_{S_r} \partial_{n_g} \phi \, dA_g = -|B|_g - \int_{\Omega} (b\sqrt{|d\phi|_g^2 + \zeta^2} + w) \, dV_g. \quad (7.21)$$

The Euclidean flux differs by $O(r^{-q})$ and has the same finite limit.

For every fixed finite $s \geq 0$, the factor u_s in Equation (7.12) is bounded and at least one, so the transformed data are complete up to B . The u -form of Lemma 7.7 gives

$$16\pi u_s^4(\mu_s - |J_s|_{g_s}) \geq 16\pi(\mu - |J|_g) + \frac{8s}{u_s}(-\Delta_g \phi - |K|_g |d\phi|_g) \geq \frac{8s}{u_s} w, \\ \theta_{+,s} = u_s^{-2} \left(\theta_+ - \frac{4s}{u_s} \right).$$

This proves the asserted signs for all finite s , without a smallness restriction.

Since $u_s - 1 = O_2(r^{-1})$, the transformed decay exponent is $\min(q, 1) > 1/2$. The exact energy transformation is

$$16\pi u_s^4 \mu_s = 16\pi \mu - 8s u_s^{-1} \Delta_g \phi.$$

Here $\Delta_g \phi = O(r^{-3-\delta})$, and the additional current term is bounded by a fixed- s constant times $|K|_g |d\phi|_g = O(r^{-3-q})$. Both are integrable in dimension three. The bounded positive conformal factor also preserves integrability in the transformed volume measure.

The leading metric change is $4s\phi\delta$, whose ADM energy flux is $-8s \int_{S_r} \partial_r \phi \, dA_\delta$ before division by 16π . All remaining metric flux terms are $O_s(r^{-q}) + O_s(r^{-1})$. Writing $\pi = K - (\text{tr}_g K)g$, one has $\pi_s = u_s^2 \pi$, so the change in its Euclidean momentum flux is $O_s(r^2 r^{-1} r^{-1-q}) = O_s(r^{-q})$. This proves Equation (7.13); Equation (7.21) gives $-L_\phi/(2\pi) > 0$. No $1/r$ coefficient expansion for ϕ is needed. Finally, the cut class is unchanged and every cut has area $\int_\Gamma u_s^4 \, dA_g$. Bounding u_s between 1 and $1 + s\|\phi\|_\infty$ and taking infima proves Equation (7.14), including coincident and disconnected cuts.

For the final assertion, take b nonnegative, compactly supported, and at least $|K|$; the proof is unchanged. The end equation is then simply $-\Delta_g \phi = w$. Once the two-derivative estimate has been established, successive annular elliptic estimates use the assumed symbol bounds for g and w to give $\phi = O_k(r^{-1})$ for every k . $\square$

7.8 Compactness of enclosing minimizers

The end replacement changes the metric on receding annuli. To compare enclosing infima, we minimize perimeter among filled inner sets and count all frontier coincident with the obstacle. We may first use competitors with bounded complementary pockets: filling those pockets deletes frontier and cannot increase perimeter. Smooth outward approximation returns to the cut class without changing its infimum. The next lemma keeps these minimizing enclosures in a fixed compact set, where local metric convergence controls their areas.

Lemma 7.8 (Enclosing areas under receding changes). *Let g_j be smooth metrics on the fixed exterior Ω , converging to g locally uniformly up to B . Suppose that in a fixed chart outside a compact set they satisfy, with constants independent of j ,*

$$c\delta \leq g_j \leq C\delta, \quad r|\partial g_j|_\delta \leq C. \quad (7.22)$$

Suppose also that each g_j has an asymptotically flat end, possibly in a different chart, and that one fixed compact enclosing competitor has g_j -areas bounded uniformly in j . Then minimizing filled enclosures exist, their boundaries lie in one fixed compact subset of Ω , and

$$a_{g_j}(B) \longrightarrow a_g(B).$$

The perimeter and smooth enclosing infima agree. Each minimizing boundary is $C^{1,1}$, and is smooth minimal away from contact with B .

Proof. For purposes of perimeter minimization only, extend the metrics through a collar of B and attach a fixed compact filling. The filling need not satisfy any constraint or curvature condition. Extend g_j so that the extensions converge uniformly on this fixed collar and filling. A filled enclosure is then a bounded set containing the prescribed inner obstacle.

For each fixed metric, sufficiently large spheres in its asymptotically flat chart have strictly positive outward mean curvature. Clipping a bounded enclosure at any such sphere cannot increase perimeter. To see this, integrate the divergence of the outward unit normal field of the sphere foliation over the part of the enclosure outside the sphere. Its divergence is positive. Its flux through the exterior boundary is at most the area of that boundary, whereas its flux on the cutting sphere is the negative of the new cutting area. Thus the discarded boundary area is at least the inserted area. Approximation gives the same statement for finite-perimeter sets. The direct method on the resulting bounded domain now gives a minimizer. The clipping observation makes this a minimizer against every bounded competitor, not merely those inside the chosen truncation.

Off the obstacle the boundary is locally perimeter minimizing. If it contains a point of radius r sufficiently large, the coordinate ball of radius $c_1 r$ about that point avoids the obstacle. After scaling this ball to unit size, Equation (7.22) gives uniform ellipticity and uniform first-derivative bounds for its area integrand. The local perimeter density estimate gives

$$\text{Area}_{g_j}(\partial F_j \cap B_{c_1 r}) \geq c_2 r^2.$$

Here $c_1, c_2 > 0$ are independent of j and r . To justify the uniform lower bound, work in a rescaled coordinate ball avoiding the obstacle, and let $v(s) = |F_j \cap B_s|_\delta$. Removing $F_j \cap B_s$ is an admissible local comparison. For almost every s , minimality and uniform metric comparability give

$$P_\delta(F_j; B_s) \leq C v'(s), \quad P_\delta(F_j \cap B_s) \leq C' v'(s).$$

The Euclidean isoperimetric inequality therefore yields $v(s)^{2/3} \leq C'' v'(s)$. At a point in the support of the minimizing frontier, both phases meet every ball. Integrating this differential inequality gives $v(s) \geq c s^3$. Filling the complement in B_s gives the same estimate for $|B_s \setminus F_j|_\delta$. The relative isoperimetric inequality now gives $P_\delta(F_j; B_s) \geq c' s^2$ on a smaller fixed ball. Returning to the original scale proves the displayed area bound. The argument also applies at all support points by taking limits of reduced-boundary points. No uniform curvature or higher metric derivative bound is used. The fixed competitor bounds the total area independently of j . Consequently the minimizing boundaries lie in a fixed compact set.

Apply [25, Regularity Theorem 1.3(iii)] with bulk integrand $f = 0$ and with U the interior of the prescribed filled obstacle. Its boundary is the smooth surface B in the extended manifold. The global minimizer also minimizes under every compactly supported change respecting U , which is the comparison class of that theorem. The ambient dimension is three, below its threshold eight. For each fixed smooth metric this gives $C^{1,1}$ regularity across contact and smoothness off contact. No bound uniform in j for this regularity is needed: the common compact support was proved by the preceding density argument, and convergence below uses only perimeter and uniform convergence of the metric. Push a minimizing boundary slightly away from its contact set by a smooth vector field agreeing there with the outward obstacle normal. In a small obstacle collar its flow increases the signed distance to the obstacle. The resulting C^1 embedded boundary is a positive distance from the obstacle. Smooth local graph approximation, within that distance, preserves enclosure and changes area by an arbitrarily small amount. This proves equality of the smooth and perimeter infima for each metric.

Enlarge the fixed compact region so that its complement is a connected coordinate tail. Each minimizing inner set is either empty or the whole tail there, since its frontier is absent. Boundedness excludes the latter, so the sets themselves also lie in a common compact set. Choose a fixed smooth auxiliary metric on this compact set. Uniform metric comparability gives uniformly bounded auxiliary perimeters, and BV compactness gives a limiting filled enclosure. The obstacle is still contained in the limit. Fill any newly bounded complementary pockets; this only removes frontier. Uniform metric convergence makes the area integrands converge uniformly relative to the auxiliary metric, so lower semicontinuity implies

$$a_g(B) \leq \liminf_j a_{g_j}(B).$$

Conversely, using a fixed enclosure with g -area arbitrarily close to $a_g(B)$ gives the reverse upper-limit inequality. $\square$

7.9 Schwarzschild reference charges

The area infimum is now stable under uniformly controlled changes on receding annuli. We next arrange those changes. The vacuum reference end must first have the original ADM charges; only after matching can we bend it to a rest slice without paying a nonvanishing constraint-repair cost.

Lemma 7.9 (A reference end with prescribed timelike charges). *Let $E > |P|$ and $m = (E^2 - |P|^2)^{1/2}$. There is a spacelike plane near spatial infinity of Schwarzschild spacetime of mass m whose future-oriented induced data (g_*, K_*) , in asymptotically Euclidean coordinates on the plane, have ADM charges (E, P) and satisfy*

$$g_* - \delta = O_k(r^{-1}), \quad K_* = O_k(r^{-2})$$

for every derivative order k .

Proof. Apply Lemma 6.2 with $n = 3$, $k = 2$, $p = 1$, and $M = m$. In static isotropic coordinates (T, y) its spacetime metric is

$$- \left(\frac{1 - m/(2r)}{1 + m/(2r)} \right)^2 dT^2 + \left(1 + \frac{m}{2r} \right)^4 \delta_{ij} dy^i dy^j, \quad r = |y|. \quad (7.23)$$

For $P \neq 0$, put $v = P/E$ and rotate the boost axis into its direction; for $P = 0$, take $v = 0$. Then $\gamma = (1 - |v|^2)^{-1/2} = E/m$, and the reference plane has charges $(m\gamma, m\gamma v) = (E, P)$. The common lemma uses the same future-normal convention for K , and its ADM prefactors become

$$\frac{1}{2k\omega_2} = \frac{1}{16\pi}, \quad \frac{1}{k\omega_2} = \frac{1}{8\pi}.$$

Thus its charges have precisely the present signs and normalization; the use of physical constraint densities here does not rescale them.

For the later bend, write the plane as $T = v \cdot y$ and set $y = Ax$, where $A = (I - v \otimes v)^{-1/2}$. The induced Minkowski metric in the x coordinates is δ . The perturbation of the metric above from Minkowski space has symbol decay of order r^{-1} on this fixed spacelike plane, and Lemma 6.2 gives the induced metric and tensor orders r^{-1} and r^{-2} through every fixed derivative order. These estimates concern the exact reference end and impose no additional derivative assumption on the original data. $\square$

7.10 Moments and compact linear corrections

The reference end has the required charges, but interpolation can violate DEC. Compact linear corrections reduce this error before the conformal repair. This is related to localized constraint deformation and its adjoint obstructions [15, 14]; those results are not being invoked as a no-parity DEC rest-end or all-cut comparison theorem.

All operators in the next two lemmas are Euclidean. For symmetric covariant tensors define

$$\mathcal{L}h = \partial_i \partial_j h_{ij} - \Delta \operatorname{tr} h, \quad (\mathcal{D}k)_i = \partial_j (k_{ij} - (\operatorname{tr} k) \delta_{ij}). \quad (7.24)$$

The formal adjoint kernel of $\mathcal{L}$ consists of affine functions; the formal adjoint kernel of $\mathcal{D}$ consists of Euclidean Killing fields. Orthogonality to these kernels is the compatibility condition for the inverses with compactly supported output below.

Lemma 7.10 (Scalar and symmetric divergence inverses with compact support). *Let $\mathcal{A} \subset \mathbb{R}^3$ be a fixed connected open annulus and let $Q \Subset \mathcal{A}$. Smooth sources supported in Q admit the following compact corrections, supported in a fixed larger compact subset of $\mathcal{A}$.*

- (i) *If $\int f = 0$ and $\int x^i f = 0$ for $i = 1, 2, 3$, there is a smooth symmetric tensor h with $\mathcal{L}h = f$ and $\|h\|_{W^{k+2,p}} \leq C_{k,p} \|f\|_{W^{k,p}}$.*
- (ii) *If $\int V \cdot Y = 0$ for every Euclidean Killing field Y , there is a smooth symmetric tensor k with $\mathcal{D}k = V$ and $\|k\|_{W^{k_0+1,p}} \leq C_{k_0,p} \|V\|_{W^{k_0,p}}$.*

The estimates hold for integers $k, k_0 \geq 0$ and $1 < p < \infty$.

Proof. Apply Lemma 6.3 in dimension three, with the prescribed source support Q and the same connected annulus. The scalar operator there is exactly $\mathcal{L}$ here, and its affine moments are the four conditions in (i). The vector operator is exactly $\mathcal{D}$; its translational and rotational moments are (ii). The inverse of trace reversal in that proof is $S \mapsto S - (\text{tr } S)\delta/(n-1)$, which becomes $S \mapsto S - (\text{tr } S)\delta/2$. In particular its symmetric correction is obtained from a skew tensor $D_{ijk} = -D_{jik}$ by

$$C_{ijk} = D_{ijk} - D_{ikj} - D_{jki}. \quad (7.25)$$

The identities $C_{ijk} = -C_{ikj}$ and $(C_{ijk} - C_{jik})/2 = D_{ijk}$ cancel the skew part without changing divergence. The two scalar inversions gain two derivatives, and the row-divergence inversion followed by that correction gains one. All supports and bounds are those of the fixed annulus. Thus the full conclusions, including linear dependence and every stated Sobolev order, are precisely the specializations of that lemma. This is an operator statement about smooth sources; the estimates available from the original weakly decaying data are verified separately in the annular replacement proof below. $\square$

For a smooth symmetric covariant tensor h on the Euclidean end and an affine scalar function f , define the Euclidean vector field

$$\mathcal{Q}_f(h)^i = f(\partial_j h_{ij} - \partial_i \text{tr}_\delta h) - (\partial_j f)h_{ij} + (\partial_i f) \text{tr}_\delta h.$$

The scalar flux paired with f is $\int_{S_R} \mathcal{Q}_f(h) \cdot n_\delta dA_\delta$; the vector flux paired with a Euclidean Killing field Y is $\int_{S_R} \pi_{ij} Y^i n_\delta^j dA_\delta$, where $\pi = K - (\text{tr}_\delta K)\delta$.

Lemma 7.11 (Integrable sources give sublinear first moments). *For data satisfying Equation (7.9) and integrable $\mu, |J|$, put $h = g - \delta$ and $\pi = K - (\text{tr}_\delta K)\delta$. Then*

$$\begin{aligned} \mathcal{L}h &= 16\pi\mu + O(r^{-2-2q}), \\ \partial_j \pi_{ij} &= 8\pi J_i + O(r^{-2-2q}). \end{aligned} \quad (7.26)$$

In particular these Euclidean sources are integrable. Their scalar boundary fluxes paired with affine functions having zero constant term, and their vector boundary fluxes paired with rotational Killing fields, are $o(R)$ on coordinate spheres of radius R .

Proof. The scalar-curvature expansion has remainder bounded by

$$C(|h| |\partial^2 h| + |\partial h|^2 + |K|^2),$$

and the replacement of the momentum constraint by Euclidean divergence has remainder bounded by $C(|h||\partial K| + |\partial h||K|)$. These are $O(r^{-2-2q})$, using exactly the derivatives in Equation (7.9). They are integrable because $2 + 2q > 3$.

For any integrable function F on the end and fixed $c > 1$,

$$\frac{1}{R} \int_{r_0 < |x| < cR} |x| |F(x)| \, dx \longrightarrow 0. \quad (7.27)$$

Indeed the contribution inside any fixed radius M tends to zero after division by R ; the remaining contribution is at most $c \int_{|x| > M} |F|$. Let $R \rightarrow \infty$ and then $M \rightarrow \infty$.

The vector field $\mathcal{Q}_f(h)$ defined above satisfies $\partial_i \mathcal{Q}_f(h)^i = f \mathcal{L}h$. For a Euclidean Killing field Y ,

$$\partial_j (\pi_{ij} Y^i) = Y^i \partial_j \pi_{ij},$$

because π is symmetric. Integrating from a fixed inner sphere and using Equation (7.27) proves the asserted $o(R)$ estimates. This argument does not assert the existence of a center-of-mass or angular-momentum limit and requires no parity condition. $\square$

Proposition 7.12 (Annular replacement with a vanishing deficit). *Suppose $E > |P|$, and let (g_*, K_*) be the reference end from Lemma 7.9, identified with the given end in the common asymptotically Euclidean coordinates. For all sufficiently large R there are smooth data $(\tilde{g}_R, \tilde{K}_R)$ equal to (g, K) on $r \leq R$ and to (g_*, K_*) on $r \geq 4R$ such that*

$$16\pi(\tilde{\mu}_R - |\tilde{J}_R|_{\tilde{g}_R}) \geq -\eta_R R^{-3}, \quad \eta_R \longrightarrow 0. \quad (7.28)$$

The possible negative part is supported in $R < r < 4R$. The metrics there are uniformly Euclidean comparable and have uniformly bounded scaled first derivatives. The construction uses no derivatives of the original data beyond those in Equation (7.9) for its quantitative estimates.

Proof. Fix $\frac{1}{2} < \beta < \min(q, 1)$ and work on the annulus $\mathcal{A} = \{1 < |y| < 4\}$ with $x = Ry$. The scaled fields are

$$g_R(y) = g(Ry), \quad k_R(y) = RK(Ry),$$

and set $g_{*,R}(y) = g_*(Ry)$, $k_{*,R}(y) = RK_*(Ry)$, $h_R = g_R - \delta$ and $h_{*,R} = g_{*,R} - \delta$. Their constraint densities are the corresponding physical densities multiplied by R^2 . Then

$$\|h_R\|_{C^2(\mathcal{A})} + \|k_R\|_{C^1(\mathcal{A})} + \|h_{*,R}\|_{C^2(\mathcal{A})} + \|k_{*,R}\|_{C^1(\mathcal{A})} \leq CR^{-\beta}. \quad (7.29)$$

Choose $\chi \in C^\infty(\mathcal{A})$ with $0 \leq \chi \leq 1$, equal to one near $|y| = 1$ and zero near $|y| = 4$, with all its derivatives supported in a fixed compact subannulus. Blend both fields by this cutoff.

Write the constraints in the unnormalized form

$$\mathcal{C}(g, k) = (\text{Scal}_g + (\text{tr}_g k)^2 - |k|_g^2, \text{div}_g(k - (\text{tr}_g k)g)).$$

Their Euclidean linear part is $(\mathcal{L}h, \mathcal{D}k)$. On fields satisfying Equation (7.29), the nonlinear remainder is bounded in supremum norm by $CR^{-2\beta}$. Thus the blended constraints equal the blended original constraint sources, plus a quadratic error and the compact commutators

$$\begin{aligned} f_R &= \mathcal{L}(\chi(h_R - h_{*,R})) - \chi \mathcal{L}(h_R - h_{*,R}), \\ V_R &= \mathcal{D}(\chi(k_R - k_{*,R})) - \chi \mathcal{D}(k_R - k_{*,R}). \end{aligned}$$

They have $C^{0,1}$ norms at most $CR^{-\beta}$. In detail the scalar commutator applied to a symmetric tensor h is

$$2\partial_i\chi(\partial_j h_{ij} - \partial_i \operatorname{tr} h) + (\partial_i \partial_j \chi)h_{ij} - (\Delta\chi) \operatorname{tr} h;$$

one derivative of this expression uses at most two derivatives of h . The vector commutator is $(\partial_j \chi)(k_{ij} - (\operatorname{tr} k)\delta_{ij})$; one derivative uses at most one derivative of k . No derivative of the complete matter sources is needed.

Every affine moment of f_R and every Killing-field moment of V_R is $o(R^{-1})$. For the constant moments, integration by parts on $\mathcal{A}$ gives R^{-1} times the difference between the physical translation fluxes on its two boundary spheres, less the integrals of the blended linear sources. The flux difference tends to zero because the reference and original ADM vectors agree. Replacing the physical momentum integrand by its Euclidean trace reversal costs $O(R^{1-2q})$, which tends to zero. By Lemma 7.11, the physical linear-source integrals on this receding annulus tend to zero as well. For a linear scalar test or a rotational vector test the corresponding factor is R^{-2} . Its physical boundary fluxes are $o(R)$ by the same lemma, while the weighted source integrals are $o(R)$ by Equation (7.27). This proves the claim.

Remove these small moments by fixed smooth bumps. Explicitly, for a basis $p_1, \dots, p_4$ of affine functions and a fixed nonnegative bump ω positive on a ball inside the annulus, the Gram matrix

$$G_{ab} = \int \omega p_a p_b$$

is positive definite. Subtract $\omega \sum_{a,b} p_a (G^{-1})_{ab} \int p_b f_R$ from f_R . For the vector moments use in exactly the same way the Gram matrix $\int \omega Y_a \cdot Y_b$ of a basis of the six Euclidean Killing fields. It too is positive definite, since a nonzero Killing field cannot vanish on an open ball. Both bump corrections have every fixed smooth norm $o(R^{-1})$.

Apply Lemma 7.10 with the negative of the moment-free commutators. Since their $W^{1,p}$ norms are $O(R^{-\beta})$, choosing $p > 3$ gives metric and tensor corrections of size $O(R^{-\beta})$ in $W^{3,p}$ and $W^{2,p}$, respectively. Sobolev embedding gives the needed $C^{2,\alpha}$ and $C^{1,\alpha}$ bounds. Their supports stay in a fixed compact subset of $\mathcal{A}$. Extend them by zero and undo the scaling. The metric remains positive definite for large R .

To check DEC, denote the original scaled sources by (A_R, B_R) , so $A_R \geq 2|B_R|_{g_R}$. The reference sources vanish. The corrected sources therefore have the form

$$(\chi A_R, \chi B_R) + o(R^{-1}) + O(R^{-2\beta})$$

in supremum norm. The norm of B_R changes by at most $CR^{-\beta}|B_R| \leq CR^{-2\beta}$ when the metric is changed, since the pointwise constraint formula gives $|B_R| \leq CR^{-\beta}$. All other nonlinear correction terms have the same quadratic bound. Because $2\beta > 1$, the scaled DEC deficit is $o(R^{-1})$. Multiplication by R^{-2} gives Equation (7.28). Outside the correction annulus the data are either the original DEC data or the exact vacuum reference data. Smoothness follows from smoothness of the original fields and of the compact inverse constructions, irrespective of the sizes of their unneeded higher derivatives. $\square$

7.11 Bending to a rest slice and repairing the deficit

The annular correction has reduced the possible DEC failure to $o(R^{-3})$ on a region of radius comparable to R . We now work inside the exact vacuum reference part. Bending makes the

distant tensor vanish; a radial conformal factor then repairs the earlier deficit at mass cost $o(1)$. Both operations must retain the uniform geometry needed by Lemma 7.8.

Lemma 7.13 (A vacuum bend with uniform geometry). *The data in Proposition 7.12 can be modified farther out, inside their exact Schwarzschild part, so that they are induced by a constant-static-time slice outside a compact set. This modification adds no constraint deficit. Write (g_R, K_R) for the resulting unscaled data. In the common end coordinates the metrics are uniformly Euclidean comparable, and throughout all transition annuli*

$$|\partial g_R|_\delta + |K_R|_\delta \leq C/r. \quad (7.30)$$

There is a smooth proper radius ρ_R on the end which equals the original coordinate radius on the gluing annulus and equals the rest isotropic radius beyond the transitions, with

$$cr \leq \rho_R \leq Cr, \quad |d\rho_R|_{g_R}^2 \geq c, \quad |\Delta_{g_R} \rho_R| \leq C/r. \quad (7.31)$$

All transition annuli lie between fixed multiples of R ; the constants can depend on the prescribed timelike ADM vector, but not on R .

Proof. Use the description $T = v \cdot y$, $y = Ax$, from Lemma 7.9. Choose $R_b = c_b R$ so large, with c_b fixed, that the region $|y| \geq R_b$ is entirely inside the exact reference part. Let ψ be a smooth function equal to one on $(-\infty, 0]$, zero on $[1, \infty)$, and taking values in $[0, 1]$. Replace the plane by the graph

$$T_R(y) = (v \cdot y) \psi \left(\frac{\log(|y|/R_b)}{L} \right). \quad (7.32)$$

Its Euclidean gradient satisfies

$$|dT_R|_\delta \leq |v|(1 + \|\psi'\|_\infty/L).$$

Choose a fixed L making the right side strictly smaller than one. The lapse and spatial coefficients in Equation (7.23) converge to their Minkowski values, so for all sufficiently large R this graph is uniformly spacelike. In fact for positive Schwarzschild mass the lapse is smaller than one and the spatial conformal factor exceeds one, which only improves this estimate on the static exterior.

The graph agrees with the original plane before R_b and with a constant-time slice after $e^L R_b$. Its first derivative is bounded, and its second and third derivatives are $O(r^{-1})$ and $O(r^{-2})$. The induced metric is uniformly comparable with δ ; differentiating its pullback formula gives the first bound in Equation (7.30). The second fundamental form uses the graph's second derivatives and ambient first derivatives, divided by a uniformly positive spacelikeness factor, and gives the other bound. These estimates remain true in the x coordinates because A is a fixed invertible linear map. The constraints vanish on the entire bent part by the Gauss–Codazzi equations in exact vacuum Schwarzschild spacetime.

It remains to make the radius precise. On the bent region keep $\rho_R = |x| = |A^{-1}y|$. Beyond the bend, where the slice is already static, write $y = r\omega$ and put $a(\omega) = |A^{-1}\omega|$. Choose R_c a sufficiently large fixed multiple of $R_b e^L$, and a cutoff χ_0 equal to one before zero and zero after one. In that static region set

$$\rho_R(r, \omega) = r \exp \left[\chi_0 \left(\frac{\log(r/R_c)}{L_0} \right) \log a(\omega) \right].$$

For fixed sufficiently large L_0 its radial derivative is bounded below by a positive constant: differentiating in r gives the positive factor ρ_R/r times $1 + L_0^{-1} \chi_0' \log a$, which is at least $1/2$.

The function and its first two Euclidean derivatives have the bounds $\rho_R \asymp r$, $|d\rho_R| \leq C$, and $|\partial^2 \rho_R| \leq C/r$. Before this interpolation, the same bounds hold for $|A^{-1}y|$, and uniform metric comparability gives the gradient lower bound. Combining these facts with Equation (7.30) proves Equation (7.31). After the interpolation, $\rho_R = r$ is precisely the static isotropic radius. All radius ratios used in the construction are fixed, independently of R . $\square$

Lemma 7.14 (A radial DEC repair of vanishing mass cost). *Consider the data obtained from Proposition 7.12 and Lemma 7.13. There is a smooth positive conformal factor u_R , constant on the unchanged inner region and tending to one at infinity, such that $(u_R^4 g_R, u_R^2 K_R)$ satisfy DEC everywhere. Moreover*

$$\|u_R - 1\|_\infty = o(R^{-1}), \quad |du_R|_{g_R} = o(R^{-2}) \quad \text{on the transition region.} \quad (7.33)$$

The final end is exactly Schwarzschild with $K = 0$ and mass $m + o(1)$. The boundary expansion retains its original weak sign.

Proof. All geometry estimates below are uniform in R . Enlarge the fixed transition region slightly so that, in terms of $t = \rho_R/R$, it lies inside an interval (a, b) with $0 < a < b < \infty$. Arrange that $t \geq b$ is in the static spherical region, and that the support of the possible deficit lies in a fixed smaller interval $I \subseteq (a, b)$. Increasing b if needed makes these assertions follow from Lemma 7.13. We take $a = 1$: the compact supports in the annular correction leave a fixed neighborhood of $t = 1$ free of deficit. The factor constructed below will consequently be constant throughout the original region $r \leq R$, and is extended by that constant over the compact core of Ω .

Let $\eta_R \rightarrow 0$ be as in Equation (7.28), replacing it by a positive majorant if necessary. We will choose $\varepsilon_R = C_0 \eta_R$ and construct

$$u_R = 1 + \frac{\varepsilon_R}{R} F_R(\rho_R/R),$$

where F_R and the relevant derivatives are bounded independently of R . Write $z_R = -F'_R$. On $[a, b + 1]$, initially define z_R by

$$z'_R = C_1 z_R + \omega_1, \quad z_R(a) = 0, \quad (7.34)$$

where ω_1 is smooth, nonnegative, zero near a , and at least one on I . Take C_1 sufficiently large, depending only on the constants in Equations (7.30) and (7.31). The solution is nonnegative and has uniform bounds on this fixed interval. For a function with $F'_R = -z_R$ the chain rule gives

$$\begin{aligned} -\Delta_{g_R} u_R - |K_R|_{g_R} |du_R|_{g_R} &= \frac{\varepsilon_R}{R^3} \left[z'_R |d\rho_R|_{g_R}^2 + R z_R \Delta_{g_R} \rho_R - R z_R |K_R|_{g_R} |d\rho_R|_{g_R} \right] \\ &\geq \frac{\varepsilon_R}{R^3} (c_1 z'_R - C_2 z_R). \end{aligned}$$

Choose C_1 so that this is nonnegative and is at least $c_1 \varepsilon_R R^{-3}$ on I .

The preceding differential inequality need not be imposed on the spherical tail. There $K_R = 0$ and the Schwarzschild radial Laplace flux coefficient is $(r + m/2)^2$. Put $c_R = m/(2R)$ and, on the still spherical interval near b , define

$$A_R(t) = (t + c_R)^2 z_R(t).$$

It has positive derivative wherever $z_R > 0$. Continue it to a positive constant on $[b + 1, \infty)$ while keeping $A'_R \geq 0$. For an explicit smooth continuation, continue the solution of Equation (7.34) to

$b + 1$, multiply the derivative of $(t + c_R)^2 z_R(t)$ by a nonnegative cutoff equal to one near b and zero near $b + 1$, and integrate with the initial value $A_R(b)$. Define $z_R = A_R(t)/(t + c_R)^2$ on this tail, and set it to zero for $t \leq a$. All joins are smooth, the functions are nonnegative, and their bounds on finite intervals are uniform in R . The constant tail value, denoted A_R^∞ , is bounded independently of R .

Now define

$$F_R(t) = \int_t^\infty z_R(s) \, ds.$$

It is nonnegative, uniformly bounded, and constant before a . The preceding construction gives a superharmonic u_R on the spherical tail, because its signed radial flux is $-\varepsilon_R A_R(t)$ and is nonincreasing. It also gives Equation (7.33). For large R , $1 \leq u_R \leq 2$. Increasing C_0 ensures

$$-\Delta_{g_R} u_R - |K_R| |du_R| \geq \frac{u_R}{8} [16\pi(|J_R|_{g_R} - \mu_R)]_+.$$

On the original inner region the conformal factor is constant and the original data satisfy DEC. Elsewhere this inequality and Lemma 7.7 prove DEC for the repaired data. Using u_R , rather than estimating $\log u_R$ separately, incorporates the gradient-square term exactly.

On the final end,

$$u_R = 1 + \frac{\varepsilon_R A_R^\infty}{r + m/2}.$$

Writing $a_R = \varepsilon_R A_R^\infty = o(1)$, one obtains the exact identity

$$u_R^4 \left(1 + \frac{m}{2r}\right)^4 \delta = \left(1 + \frac{m/2 + a_R}{r}\right)^4 \delta. \quad (7.35)$$

Thus the final mass is $m + 2a_R = m + o(1)$ and the momentum is zero. Multiplication of K_R by u_R^2 preserves its compact support. On the inner boundary, where u_R is constant, the expansion is simply multiplied by u_R^{-2} , so its weak sign is preserved. $\square$

7.12 Completion of the end reduction

The preceding construction has replaced the end while preserving DEC and the boundary expansion sign. A final strict conformal change puts the data in the class used for the elliptic deformation.

Proof of Proposition 7.4. Apply Proposition 7.12, Lemma 7.13, and Lemma 7.14 along any sequence $R_j \rightarrow \infty$. The resulting DEC exteriors have compactly supported second fundamental form and an exact rest Schwarzschild end with energy tending to m . They agree with the original data on expanding compact subsets except for a constant conformal factor tending to one. Hence they converge smoothly on each fixed compact subset. Their boundary expansion is weakly negative.

Apply the last part of Lemma 7.5 separately to each of these fixed exteriors, using the same chosen $0 < \delta < 1$ and a compactly supported drift coefficient. Choose its positive parameter s in Equation (7.12) small enough that the energy change, $\|\log(1 + s\phi)\|_\infty$, and the supremum of its first derivative multiplied by the fixed original coordinate radius are at most $1/j$. Also require the change in the first j smooth norms on the first j members of a compact exhaustion to be at most $1/j$. All these quantities are finite for the fixed j -th exterior, so these choices are possible. No estimate uniform in the receding bending radius is required for this final strict conformal deformation.

The resulting data have pointwise strict DEC and strict future outer trapping. The quantitative end gap is at least $c_j r^{-3-\delta}$. On its compact complement the continuous positive function $(\mu_j - |J_j|_{g_j})\rho^{3+\delta}$ has a positive minimum. Replacing c_j by the smaller of this minimum and the end constant proves the global weighted assertion in (i). The strict conformal factor has symbol decay of all orders and does not enlarge the support of K . On the far Schwarzschild end the drift vanishes, so $-\Delta\phi = w = r^{-3-\delta}$. Since the rest Schwarzschild metric has zero scalar curvature and $K = 0$ there, the exact u -formula gives

$$R_{g_j} = 8s(1 + s\phi)^{-5}r^{-3-\delta}.$$

This proves the scalar-curvature bound in (ii). The Ricci bound follows from the two-derivative metric decay. The energy change tends to zero and the momentum change is zero. The prescribed smallness gives smooth local convergence, proving (ii) and the local convergence and charge assertions in (iii).

The metrics before strict conformal deformation satisfy the uniform comparisons and scaled first-derivative bounds of Lemma 7.8, in the fixed original end chart: the reference plane, the bend, and the radius interpolation involve only fixed linear maps and fixed radius ratios. The radial repair preserves these bounds. The final strict conformal deformation parameters were chosen to preserve them as well. That lemma therefore gives the convergence of enclosing infima.

The energy and enclosing-area limits reduce Theorem 7.2 to the class in (i)–(ii). No numerical inequality was used in constructing these data. $\square$

7.13 Complete data with finitely many ends

The one-ended inequalities have now been proved. We pass to a complete boundaryless initial-data manifold. The ambient enclosing class allows other ends to remain on the chosen side. We first isolate the chosen end to use the exterior theorem, and then compare the two area infima in the direction required for the conclusion.

Definition 7.15 (Admissible boundary and ambient enclosing area). Let (M, g, K) be a smooth three-dimensional initial data set without boundary, with M connected, and fix one asymptotically flat end e . An admissible boundary for e is a nonempty compact smooth embedded two-sided surface $S = \partial D^+$, where D^+ is a smooth open region containing the entire sufficiently distant part of e . The surface and the region may be disconnected, and D^+ may contain other ends. For the normal ν pointing into D^+ , require

$$H_S + \text{tr}_S K \leq 0.$$

Its minimum enclosing area is

$$A_{\min}(S) = \inf \left\{ \text{Area}_g(\partial D'^+) : \begin{array}{l} D'^+ \subseteq D^+ \text{ is a smooth open region containing} \\ \text{the entire sufficiently distant part of } e, \\ \partial D'^+ \text{ is compact, smooth, and embedded} \end{array} \right\}. \quad (7.36)$$

The full boundary area is counted, including portions coincident with S . Competitor regions D'^+ may be disconnected and may retain other ends. Their boundaries need be neither trapped nor minimal. The definition imposes no sign condition on the inward null expansion.

Theorem 7.16 (Spacetime Penrose inequality for complete initial data). *Let (M, g, K) be a connected orientable smooth three-dimensional initial data set without boundary, with g complete.*

Suppose that outside a compact set M has finitely many asymptotically flat ends. On each end, in its asymptotic coordinates, assume

$$g - \delta = O_2(r^{-q}), \quad K = O_1(r^{-1-q}), \quad q > \frac{1}{2},$$

and finite ADM energy and momentum limits with the normalizations (7.4)–(7.5). The decay exponents may differ between ends. Assume that the physical constraint densities (7.1) obey $\mu \geq |J|_g$ and that both μ and $|J|_g$ are integrable on M .

For every chosen end and every admissible boundary S in Definition 7.15, its ADM vector (E, P) satisfies

$$E > |P|_\delta, \quad \sqrt{E^2 - |P|_\delta^2} \geq \sqrt{\frac{A_{\min}(S)}{16\pi}}. \quad (7.37)$$

Proof. Let D_0 be the connected component of D^+ containing the distant chosen end. Its boundary is a union of components of S . It is nonempty: otherwise D_0 would be both open and closed in the connected manifold M , and hence equal to M , contradicting $S \neq \emptyset$. The closure $X_0 = \overline{D_0}$ is therefore a smooth manifold with nonempty compact boundary. Since S is compact, each sufficiently distant end of M lies either entirely in D_0 or entirely outside it. Thus X_0 has finitely many ends, precisely those original end tails contained in D_0 .

For every end of X_0 other than the chosen one, remove a sufficiently distant open coordinate tail. The tails can be chosen disjoint and disjoint from S . Denote the remainder by Ω . It is connected: a path segment entering one of the removed tails can be replaced by a path on the connected coordinate sphere bounding that tail. Consequently Ω is a one-ended smooth exterior with compact boundary

$$B = \partial D_0 \cup \bigcup_{\text{other ends of } X_0} S_{R_i}.$$

This boundary is nonempty because ∂D_0 is nonempty.

The set Ω is closed in the complete manifold (M, g) , and it is complete for its intrinsic distance with B included. Indeed, an intrinsic Cauchy sequence is ambient Cauchy and has its ambient limit in Ω . At an interior limit point the two distances are locally comparable. At a boundary limit point, a smooth boundary chart identifies Ω locally with a half-ball and gives the same comparison, so the sequence converges intrinsically as well. Restricting the data preserves DEC and integrability. The chosen end is unchanged, so its decay and ADM vector remain the original ones.

On ∂D_0 the normal into Ω is the normal into D^+ , and the future expansion is nonpositive. On an added sphere S_{R_i} , the normal into Ω points toward decreasing radius in the removed end. The differentiated asymptotic assumptions give

$$H_{S_{R_i}} = -\frac{2}{R_i} + O(R_i^{-1-q_i}), \quad \text{tr}_{S_{R_i}} K = O(R_i^{-1-q_i}).$$

Choose each R_i large enough that their sum is negative. Every component of B is then weakly future outer trapped for the normal into Ω . Theorem 7.6 applies and yields

$$E > |P|_\delta, \quad \sqrt{E^2 - |P|_\delta^2} \geq \sqrt{\frac{a_g(B)}{16\pi}}. \quad (7.38)$$

It remains to compare the enclosing classes. We claim

$$a_g(B) \geq A_{\min}(S). \quad (7.39)$$

Take first a smooth enclosing cut Γ contained in the interior of Ω . The connected component on its distant chosen-end side is a smooth open region $D'^+ \subset D_0 \subset D^+$. Its boundary consists of components of Γ , so it is compact and smooth and has area at most $\text{Area}_g(\Gamma)$. It is therefore an admissible competitor in (7.36). For a cut that meets or coincides with B , a smooth collar push into the interior, with the obstacle side filled, gives enclosing cuts whose areas converge to its full area. This applies also to contact with the added coordinate spheres. Hence every enclosing hypersurface of Ω has area at least $A_{\min}(S)$, proving (7.39) after taking the infimum. The discarded components of D^+ do not obstruct this comparison: the ambient definition permits the chosen-end component alone as a competitor. Combining the area comparison with (7.38) proves the theorem. $\square$

The auxiliary exterior excludes the other distant ends, whereas the ambient competitor class allows them. The proof uses only the one-sided comparison (7.39). Rigidity is a separate question about the original data; it requires its own argument rather than equality transport through these conformal changes or end truncations.

8 Weak asymptotics in four dimensions

We prove a lower bound for the ADM rest mass in terms of the least three-volume needed to separate a chosen asymptotically flat end from the compact weakly outer trapped boundary and the other ends. The task is to bring two derivatives of the metric and one derivative of the second fundamental form within the scope of Theorem 2.3. We first replace the selected end by a rest end while controlling every cut, then make the dominant energy and boundary inequalities strict. Theorem 2.3 is applied to each fixed resulting datum; only its numerical conclusion is passed to the limit.

8.1 Initial data and enclosing cuts

Four is the spatial dimension; the associated spacetime dimension is five, and a hypersurface area is a three-dimensional Riemannian volume. Here $\omega_3 = |S^3| = 2\pi^2$, and we write $\tau = \text{tr}_g K$. Specializing the constraint convention already fixed in the paper gives

$$2\mu = R_g + (\text{tr}_g K)^2 - |K|_g^2, \quad J_i = \nabla^j (K_{ij} - (\text{tr}_g K)g_{ij}).$$

Here $R_g = \text{Scal}_g$ denotes scalar curvature and ∇ is the Levi-Civita connection of g . The cosmological constant is zero. We use $\Delta_g = \text{div}_g \text{grad}_g$. For a hypersurface with specified unit normal ν , write

$$H = \text{div}_{\tan} \nu, \quad \text{tr}_{\tan} K = (g^{ij} - \nu^i \nu^j) K_{ij}, \quad \Theta_K = H + \text{tr}_{\tan} K.$$

At an inner boundary the normal points into the exterior. In an auxiliary trapped-region construction it points out of the bounded region.

We retain the differentiated-decay notation of Definition 1.1. On an asymptotically flat end the ADM normalization is

$$E = \frac{1}{6\omega_3} \lim_{R \rightarrow \infty} \int_{|x|=R} (\partial_j g_{ij} - \partial_i g_{jj}) n_\delta^i \, dA_\delta, \quad (8.1)$$

$$P_i = \frac{1}{3\omega_3} \lim_{R \rightarrow \infty} \int_{|x|=R} (K_{ij} - \tau g_{ij}) n_\delta^j \, dA_\delta. \quad (8.2)$$

Here n_δ and dA_δ are Euclidean. Whenever a charge is used below, its stated limit is required to exist.

Definition 8.1 (Enclosing cuts). Let M have nonempty compact boundary S and finitely many asymptotically flat ends, and fix an end e . An admissible outer domain D_e is a connected smooth codimension-zero submanifold with boundary, closed as a subset of M , such that its manifold interior lies in $\text{int } M$, it contains the entire sufficiently distant part of e , and it contains no sufficiently distant part of any other end. Its entire intrinsic manifold boundary $\Gamma = \partial_{\text{man}} D_e$ is required to be compact. The enclosing volume is

$$A_e(S) = \inf_{D_e} |\partial_{\text{man}} D_e|_g.$$

All frontier coincident with S is counted. The cut may be disconnected. In the one-ended case $D_e = M$ is allowed and has intrinsic boundary S . In that case write $A_*(M, g) = A_{\min, g}(S)$ for this infimum, and abbreviate it by $A_*(g)$ when M is fixed or by A_* when both are fixed.

We use the infimum over these smooth cuts without requiring attainment. The entire original boundary and the distant parts of all other ends are excluded from the interior of the selected outer domain. This differs from the three-dimensional complete-data class, which allows disconnected outer regions retaining other ends.

Theorem 8.2 (The four-dimensional enclosing-area inequality). *Let (M^4, g, K) be smooth and connected, with M orientable, K an arbitrary smooth symmetric covariant two-tensor, and $S = \partial M$ nonempty, compact, and smooth. Suppose g is complete as a metric space with S included. Outside a compact set let M have finitely many, and at least one, ends diffeomorphic to the complement of a closed ball in $\mathbb{R}^4$. Assume*

$$\mu \geq |J|_g, \quad \mu, |J|_g \in L^1(M, dV_g),$$

and, on every end for a common $q > 1$,

$$g - \delta = O_2(r^{-q}), \quad K = O_1(r^{-1-q}),$$

with finite energy and momentum limits (8.1)–(8.2). On every component of S assume $\Theta_K \leq 0$, with normal into M .

Then for each end e ,

$$E_e \geq \sqrt{|P_e|_\delta^2 + \frac{1}{4} \left(\frac{A_e(S)}{\omega_3} \right)^{4/3}}. \quad (8.3)$$

In particular $E_e > |P_e|_\delta$.

Equivalently, the invariant mass obeys

$$\sqrt{E_e^2 - |P_e|_\delta^2} \geq \frac{1}{2} \left(\frac{A_e(S)}{\omega_3} \right)^{2/3}. \quad (8.4)$$

The strict timelikeness is a conclusion: Lemma 8.3 establishes $A_e(S) > 0$, and the proof includes the non-timelike case. There is no extra decay assumption on $\tau = \text{tr}_g K$. All original boundary components use the same future outer trapping convention.

The coefficient is sharp. On the time-symmetric Schwarzschild–Tangherlini exterior [44],

$$g_m = \left(1 + \frac{m}{2r^2}\right)^2 \delta, \quad K = 0, \quad r \geq \sqrt{m/2},$$

the ADM energy is m , the momentum is zero, and the boundary has three-volume $\omega_3(2m)^{3/2}$. The areal radius $r + m/(2r)$ is nondecreasing on this exterior. Let α be the pullback of the boundary’s area form by radial projection. It is closed and has comass at most one, because the areal radius is nondecreasing. On the compact region between any full enclosing cut Γ and a distant sphere, Stokes’ theorem gives $\int_\Gamma \alpha = \omega_3(2m)^{3/2}$ with the orientation toward the end. This identity includes all components of Γ ; boundary contact can be handled by a collar push and area convergence. Consequently $|\Gamma| \geq \omega_3(2m)^{3/2}$, and the boundary attains that value. The minimum enclosing volume is the boundary volume, and (8.3) is an equality for this family. The theorem concerns the numerical bound. Classification at equality requires a separate argument on the original data.

8.2 The exact cut class and removal of other ends

Lemma 8.3 (Cut reduction and positivity). *Under the hypotheses of Theorem 8.2, the quantity $A_e(S)$ is finite and strictly positive for every end e . Truncating all other ends at sufficiently large coordinate spheres gives a connected, complete one-ended exterior M_T with compact smooth boundary S_T . All its boundary components have future nonpositive expansion with the normal into M_T . Its selected-end charges are the original charges, and*

$$A_*(M_T, g) \geq A_e(S), \quad \lim_{T \rightarrow \infty} A_*(M_T, g) = A_e(S). \quad (8.5)$$

The common parameter T in the limit can be replaced by any exhaustion in which every unwanted truncation radius tends to infinity.

Proof. A sufficiently distant sphere in the chosen end bounds an admissible outer domain, so the infimum is finite. To obtain a positive lower bound, choose an embedded arc from an open disk in S to the chosen end, and continue it along a straight coordinate ray. The compact part of the arc can be chosen embedded and disjoint from the boundary except at its initial point. A boundary collar and the tubular neighborhood theorem give a proper tube with coordinates

$$[0, \infty) \times B^3 \longrightarrow M.$$

Its initial section lies in S ; its far part is a Euclidean tube of fixed transverse coordinate size. Choose a smooth nonnegative function η compactly supported in B^3 , with integral one, and define in this tube

$$\alpha = \eta(z) \, dz^1 \wedge dz^2 \wedge dz^3.$$

Extend α by zero across the lateral tube boundary. It is a smooth closed three-form on M . Its comass, meaning the supremum of its absolute value on unit simple three-vectors, has a finite bound C_α : the initial part is compact, and the far metric is uniformly comparable with the Euclidean metric. A sufficiently distant selected-end sphere has α -integral one, after fixing orientation.

Let D_e be any admissible outer domain with cut Γ . Cut off its selected-end tail at a sphere beyond Γ . The resulting domain is compact, since D_e contains no distant part of another end. Its

boundary is the distant sphere together with the *entire* intrinsic boundary Γ , with the boundary orientations. Stokes' theorem gives

$$\left| \int_{\Gamma} \alpha \right| = 1, \quad |\Gamma|_g \geq C_{\alpha}^{-1}.$$

This proof applies to disconnected cuts and to frontier on S . In particular, it proves positivity before any end is truncated.

On an unwanted end, the normal of the truncation sphere that points into the retained manifold is the inward radial normal. The given decay therefore gives

$$H = -3T^{-1} + O(T^{-1-q}), \quad \text{tr}_{S_T} K = O(T^{-1-q}).$$

The new boundary has strictly negative future expansion for large T . The other boundary components are the original ones. Removing the open tails leaves a connected manifold: a path entering a removed tail can have that portion replaced by a path along its connected spherical cross-section. The retained manifold is closed in M . It is complete for its intrinsic metric as well. Indeed, an intrinsic Cauchy sequence is ambient Cauchy and has a limit in the retained set; near a smooth boundary or an interior point, the intrinsic and ambient local distances are comparable in a half-ball or ball chart.

Every admissible outer domain in M_T , viewed as a submanifold of M , is an admissible outer domain for e . A part of its frontier on a truncation sphere becomes a hypersurface in the interior of M , and remains part of its intrinsic boundary. This proves the first inequality in (8.5). Conversely, a fixed original admissible domain has compact frontier and contains no tail of any unwanted end. It is consequently contained in M_T for all sufficiently large truncation radii and is then an admissible domain there. Applying this observation to cuts with area within any prescribed positive error of $A_e(S)$ proves the limit. The selected-end data have not been changed, so neither selected-end charge changes. $\square$

We may thus work on one-ended data until the final transfer. The boundary is always included. No finite-perimeter relaxation or minimizing cut is needed in this section.

8.3 Preparation statements and deduction of the inequality

After the cut reduction, three statements suffice for the numerical argument. We state their full outputs before constructing them. The rest-end replacement is proved in Section 8.7; the strict conformal deformation is proved in Section 8.8.

Proposition 8.4 (Rest-end replacement). *Let (M, g, K) satisfy the one-ended hypotheses of Theorem 8.2, and suppose $E > |P|$. Put $m = (E^2 - |P|^2)^{1/2}$. There is a sequence of smooth data (g_R, K_R) on the same manifold, complete with the same compact boundary, satisfying the dominant energy condition and the same future boundary inequality, such that:*

- (i) K_R is compactly supported, and the ultimate end is an exact static Schwarzschild–Tangherlini spatial end;
- (ii) its energy is $m_R = m + o(1)$, its momentum is zero, and both constraint densities are integrable;
- (iii) $A_*(g_R) \geq (1 - o(1))A_*(g)$.

Every error here tends to zero as $R \rightarrow \infty$ for the fixed original data and fixed timelike vector. No uniformity as $E - |P|$ tends to zero is asserted or needed.

Lemma 8.5 (Strict conformal deformation of a fixed rest end). *Let (M, g, K) be a smooth connected four-dimensional exterior, complete with its nonempty compact smooth boundary included, with one Euclidean coordinate end and compact complement. Suppose $\mu \geq |J|_g$ and $\Theta_K \leq 0$ on every boundary component, where the boundary normal ν points into M . Assume that K has compact support and that, outside a compact set,*

$$g = \left(1 + \frac{c}{r^2}\right)^2 \delta$$

in the end coordinates, for a constant c . Fix $0 < \delta_1 < 1$, and extend the coordinate radius to a positive smooth function r on M .

There is a nonnegative smooth function ϕ , with $\phi = O_k(r^{-2})$ for every fixed derivative order k , such that, for all sufficiently small $s > 0$, the data

$$(g_s, K_s) = (e^{2s\phi}g, e^{s\phi}K)$$

satisfy

$$\Theta_{K_s} < 0, \quad \mu_s - |J_s|_{g_s} \geq c_s r^{-4-\delta_1} \quad \text{on } M, \quad c_s > 0. \quad (8.6)$$

They are complete, have compactly supported second fundamental form, integrable constraint densities, zero ADM momentum and finite ADM energy. Their end satisfies

$$g_s - \delta = O_k(r^{-2}) \quad \text{for every fixed } k, \quad R_{g_s} = O(r^{-4-\delta_1}). \quad (8.7)$$

For the enclosing-hypersurface infimum on this fixed exterior,

$$E_{g_s} \longrightarrow E_g, \quad A_*(g_s) \longrightarrow A_*(g) \quad \text{as } s \downarrow 0. \quad (8.8)$$

The last input applies to a one-ended datum without a timelike assumption. Its proof is in Section 8.5.

Lemma 8.6 (Increasing energy while fixing momentum). *For any one-ended data in Theorem 8.2 and every $\lambda > 0$, there is a smooth positive function $u_\lambda \geq 1$, constant near S , for which $(g_\lambda, K_\lambda) = (u_\lambda^2 g, u_\lambda K)$ preserves those hypotheses, with decay exponent $\min(q, 2) > 1$, and*

$$E_\lambda = E + 2\lambda, \quad P_\lambda = P, \quad A_*(g_\lambda) \geq A_*(g).$$

8.4 Returning to the original end

The proper radial comparison protects the original enclosing area during rest-end replacement. The strict conformal deformation then changes the cut infimum by a factor tending to one at each fixed replacement radius. We apply the inequality of Theorem 2.3 after both constructions, remove strict conformal deformation at that fixed radius, and finally send the replacement radius to infinity.

Proof of Theorem 8.2. First let the original exterior have one end and $E > |P|$. Set $m = (E^2 - |P|^2)^{1/2} > 0$. Proposition 8.4 gives data (g_R, K_R) with zero momentum and energy $m_R \rightarrow m$. Its enclosing-area comparison is

$$A_*(g_R) \geq (1 + \epsilon_R)^{-3/2} A_*(g), \quad \epsilon_R \longrightarrow 0. \quad (8.9)$$

Fix one sufficiently large replacement radius R . For any fixed $0 < \delta_1 < 1$, extend this datum's rest-end radius smoothly and positively as ρ_R . Apply Lemma 8.5 to (g_R, K_R) with its radius function $r = \rho_R$, denote the resulting function by ϕ_R , and set

$$g_{R,s} = e^{2s\phi_R} g_R, \quad K_{R,s} = e^{s\phi_R} K_R.$$

These data satisfy Definition 2.1 for all sufficiently small $s > 0$. Write $\mu_{R,s}, J_{R,s}, E_{R,s}$ for their geometric constraint densities and ADM energy. Indeed it is the same smooth connected orientable one-ended exterior, complete with its compact boundary included and with compact complement of its coordinate end. It has finite ADM energy and a compactly supported tensor, so its ADM momentum is zero. On the end, $g_{R,s} - \delta = O_d(r^{-2})$ for every fixed derivative order d , its scalar curvature is $O(r^{-4-\delta_1})$, and its constraint densities are integrable. The boundary is strictly future outer trapped, and the global margin is

$$\mu_{R,s} - |J_{R,s}|_{g_{R,s}} \geq c_{R,s} \rho_R^{-4-\delta_1}, \quad c_{R,s} > 0.$$

These are the geometric and decay conditions of Definition 2.1, with $n = 4$ and $\beta = \delta_1$.

The area condition follows along the same comparisons. In one end, Definition 8.1 is exactly the enclosing class in Definition 1.2: the outer domain is connected and closed, its interior lies in the open exterior, and its entire compact intrinsic boundary is counted, including every component and all coincidence with the original boundary. Lemma 8.3 gives $A_*(g) > 0$; Equation (8.9) and strict conformal deformation give $A_*(g_{R,s}) \geq A_*(g_R) > 0$. The constraint and ADM normalizations also agree: $2\mu = R + (\text{tr } K)^2 - |K|^2$, $J = \text{div}(K - (\text{tr } K)g)$, and the energy and momentum factors are $1/(6\omega_3)$ and $1/(3\omega_3)$. No rescaling is made.

Theorem 2.3 therefore yields, at every such fixed pair (R, s) ,

$$E_{R,s} \geq \frac{1}{2} \left(\frac{A_*(g_{R,s})}{\omega_3} \right)^{2/3}. \quad (8.10)$$

This is the completed numerical theorem: its elliptic exhaustion and its limits $N \rightarrow \infty$ and $\epsilon \downarrow 0$ have already been taken for the fixed input datum. None of those estimates is required to be uniform in R or s .

Now let $s \downarrow 0$ with R fixed. The strict conformal deformation gives both $E_{R,s} \rightarrow m_R$ and $A_*(g_{R,s}) \rightarrow A_*(g_R)$, so

$$m_R \geq \frac{1}{2} \left(\frac{A_*(g_R)}{\omega_3} \right)^{2/3} \geq \frac{1}{2} (1 + \epsilon_R)^{-1} \left(\frac{A_*(g)}{\omega_3} \right)^{2/3}.$$

Only at this point let $R \rightarrow \infty$. The result is

$$\sqrt{E^2 - |P|^2} \geq \frac{1}{2} \left(\frac{A_*(g)}{\omega_3} \right)^{2/3} \quad (E > |P|). \quad (8.11)$$

Full convergence of $A_*(g_R)$ has not been used or asserted.

It remains to remove the timelike assumption. If instead $E \leq |P|$, take any sequence $a_j > 0$ tending to zero and set

$$\lambda_j = \frac{|P| + a_j - E}{2} > 0.$$

Lemma 8.6 gives, separately for each j , data of energy $|P| + a_j$, momentum P , and enclosing infimum at least $A_*(g)$. Their charges are future timelike, so (8.11) gives

$$\sqrt{(|P| + a_j)^2 - |P|^2} \geq \frac{1}{2} \left(\frac{A_*(g)}{\omega_3} \right)^{2/3} > 0.$$

The right side is fixed and positive by Lemma 8.3, while the left side tends to zero. This contradiction includes zero, negative, and null energy cases. No preparation geometry is passed to a limit as the boost degenerates: the timelike numerical result was applied afresh to each datum in this conformal family. Thus $E > |P|$ and (8.11) hold for every one-ended original datum in the theorem.

Finally let the original manifold have finitely many ends and fix one end e . Truncate every other end at a sufficiently large sphere as in Lemma 8.3. The retained manifold is a complete connected one-ended exterior. Its added boundary components are strictly future outer trapped, its selected-end charges are still (E_e, P_e) , and its enclosing-area infimum is at least $A_e(S)$. Applying the one-ended result gives

$$E_e \geq \sqrt{|P_e|^2 + \frac{1}{4} \left(\frac{A_e(S)}{\omega_3} \right)^{4/3}}.$$

Positivity of $A_e(S)$ was proved before any truncation. This gives both the asserted inequality and strict future timelikeness at e . $\square$

The one-ended cases of Theorems 7.6 and 8.2 together prove Theorem 1.5.

8.5 Conformal changes that increase energy

Lemma 8.7 (Conformal constraint and boundary formulas). *Let g be any smooth four-dimensional Riemannian metric, K a smooth symmetric covariant two-tensor, and $u > 0$ a smooth function. Set $g_u = u^2 g$ and $K_u = uK$, with the geometric constraint densities defined at the start of this section. As identities of scalar functions and covectors, respectively,*

$$u^2 \mu_u = \mu - 3u^{-1} \Delta_g u, \tag{8.12}$$

$$u J_u = J + 3K(\text{grad}_g \log u, \cdot). \tag{8.13}$$

Consequently,

$$u^2 (\mu_u - |J_u|_{g_u}) \geq \mu - |J|_g + 3u^{-1} (-\Delta_g u - |K|_g |du|_g). \tag{8.14}$$

For any consistently oriented hypersurface,

$$\Theta_{K_u} = u^{-1} (\Theta_K + 3\partial_\nu \log u). \tag{8.15}$$

Proof. Use Equation (6.2) with $n = 4$, $k = 3$, $p = 2$, and $s = \log u$. The constraint normalization is unchanged. At zero vector argument its first identity gives $u^2 \mu_u = \mu - 3\Delta \log u - 3|d \log u|^2 = \mu - 3u^{-1} \Delta u$. The covector formula proved with that identity gives $u J_u = J + 3K(\nabla \log u, \cdot)$; taking its g_u norm adds another factor u^{-1} . The triangle inequality then gives Equation (8.14), equivalently the $U = u$ specialization of Equation (6.3). The boundary formula reads $\Theta_{K_u} = u^{-1} (\Theta_K + 3\partial_\nu \log u)$, with the same chosen normal. The identities require smooth fields and $u > 0$ only, so this application does not strengthen the original O_2/O_1 decay hypotheses. $\square$

Proof of Lemma 8.6. Choose $0 < \xi < \min(q, 1)$. For $T(r) = r^{-2} - r^{-2-\xi}$, at sufficiently large radius,

$$T' < 0, \quad -\Delta_g T - |K|_g dT|_g = \xi(2 + \xi)r^{-4-\xi} + O(r^{-4-q}) > 0.$$

Take a nondecreasing smooth switch β from zero to one in this region, constant on neighborhoods of its endpoints, and set

$$G(r) = - \int_r^\infty \beta(s) T'(s) ds.$$

Extend G constantly throughout the interior. Then $G \geq 0$, it equals T sufficiently far out, and

$$-\Delta_g G - |K|_g dG|_g = \beta(-\Delta_g T - |K|_g dT|_g) - \beta' T' dr|_g^2 \geq 0.$$

Use $u = 1 + \lambda G$ in Lemma 8.7. The dominant energy condition is preserved, as is the boundary expansion sign. The new densities are integrable: outside a compact set the additional terms in (8.12)–(8.13) are bounded by constants, depending on λ , times $r^{-4-\xi} + r^{-4-q}$. The new falloff exponent can be taken to be $\min(q, 2) > 1$: the added factor is $O_2(r^{-2})$ and its multiplication into K uses only one derivative. For each fixed finite λ , $1 \leq u \leq 1 + \lambda \|G\|_\infty < \infty$, so $g \leq g_\lambda \leq (1 + \lambda \|G\|_\infty)^2 g$. This preserves completeness including S on the same one-ended manifold. All assertions here are for that fixed λ .

For the charges, the leading change in the metric flux numerator is $-6\partial_i(\lambda G)$. Products involving $g - \delta$, $u - 1$, or their first derivatives have vanishing limiting flux. Hence (8.1) gives

$$E_\lambda - E = -\frac{\lambda}{\omega_3} \lim_{r \rightarrow \infty} \int_{S_r} \partial_r G dA_\delta = 2\lambda.$$

The transformed momentum tensor is exactly $u(K - \tau g)$. Its additional flux is $O(r^{-q})$, so $P_\lambda = P$ by (8.2). Since $u \geq 1$, every cut has at least its original area, proving the cut comparison. $\square$

This conformal change will be used only after proving the inequality for $E > |P|$. It then excludes all remaining ADM vectors; no preliminary strict timelikeness hypothesis will remain.

8.6 Matching the charges before making a rest end

A rest end must have energy close to $\sqrt{E^2 - |P|^2}$, not to E . We therefore first attach a vacuum reference plane carrying exactly (E, P) , and only then bend that plane to a static slice. Compact linear corrections make the interpolation deficit small; a positive conformal repair restores DEC at vanishing mass cost.

The reference plane

Lemma 8.8 (Explicit reference-plane charges). *For any $E > |P|$, put $m = (E^2 - |P|^2)^{1/2}$. A spacelike plane in the Schwarzschild–Tangherlini spacetime of mass m , sufficiently far from its central region, has induced vacuum data (g_*, K_*) with charges exactly (E, P) , and*

$$g_* - \delta = O_k(r^{-2}), \quad K_* = O_k(r^{-3}) \quad \text{for every integer } k \geq 0.$$

Here K_* is one half the metric rate in the future normal direction.

Proof. Use Lemma 6.2 with $n = 4$, $k = 3$, $p = 2$, and $M = m$. Its exact isotropic spacetime metric is

$$\mathbf{g} = - \left(\frac{1 - m/(2|y|^2)}{1 + m/(2|y|^2)} \right)^2 db^2 + \left(1 + \frac{m}{2|y|^2} \right)^2 \sum_{i=1}^4 (dy^i)^2. \quad (8.16)$$

For $P \neq 0$, rotate the first axis into the direction of P ; for $P = 0$, choose any first axis. Put $v = |P|/E$, $\gamma = (1 - v^2)^{-1/2}$. The Lorentz plane in that lemma has $E_* = m\gamma = E$ and $P_{*1} = m\gamma v = |P|$, with all transverse momenta zero. These are the present factors $1/(6\omega_3)$ and $1/(3\omega_3)$: in particular no mass is rescaled. On $T = 0$, use the Lorentz coordinates x from Equation (6.4) and put $\rho = (\gamma^2(x^1)^2 + \sum_{a=2}^4 (x^a)^2)^{1/2}$. The leading energy flux is $6m\gamma^2 x_i \rho^{-4}$, the leading momentum tensor is the $k = 3$ instance of (6.7), and the angular integral in (6.8) is ω_3/γ . Thus the sign and normalization are the same ones used here. The second form uses the same future-normal convention. Finally the metric and tensor remainders in that lemma are $O_d(r^{-4})$ and $O_d(r^{-5})$, with leading orders r^{-2} and r^{-3} , through every fixed derivative order. Gauss–Codazzi in the vacuum metric (8.16) proves that the reference data are vacuum. $\square$

Compact inverses for the linear constraints

As in localized constraint deformation [15, 14], the adjoint kernels determine which source terms can be removed by compactly supported corrections. We remove that part of the interpolation error below. The remaining error is repaired by a conformal change, after which we compare the areas of all admissible cuts.

On Euclidean four-space write

$$\mathcal{L}b = \partial_i \partial_j (b_{ij} - (\text{tr}_\delta b) \delta_{ij}), \quad (\mathcal{D}k)_i = \partial_j (k_{ij} - (\text{tr}_\delta k) \delta_{ij}). \quad (8.17)$$

These are the linearizations at $(\delta, 0)$ of 2μ and J . The next lemma identifies every compatibility condition needed for a compact correction.

Lemma 8.9 (Compact linear constraint corrections). *Fix a compact subannulus U_0 of a bounded connected Euclidean annulus U . If f and F are smooth and supported in U_0 , and*

$$\begin{aligned} \int f \phi \, dz &= 0 \quad \text{for every affine } \phi, \\ \int F_i Y^i \, dz &= 0 \quad \text{for every Euclidean Killing field } Y, \end{aligned} \quad (8.18)$$

there are symmetric smooth tensors b, k , supported in a fixed compact subset of U , with $\mathcal{L}b = f$, $\mathcal{D}k = F$. For every integer $j \geq 0$ and every $1 < p_1 < \infty$, they can be chosen linearly in the sources so that

$$\|b\|_{W^{j+2}, p_1} \leq C \|f\|_{W^{j, p_1}}, \quad \|k\|_{W^{j+1}, p_1} \leq C \|F\|_{W^{j, p_1}}. \quad (8.19)$$

The constant depends only on U_0, U, j, p_1 and fixed localization choices. The moment conditions are also necessary for compactly supported solutions.

Proof. Lemma 6.3 applies with $n = 4$ on the given connected annulus. Its scalar operator $\mathcal{S}$ is $\mathcal{L}$, and its vector operator $\mathcal{D}$ is $\mathcal{D}$. Its affine and Euclidean Killing moments are exactly (8.18). For a symmetric covariant tensor S , the inverse trace reversal specializes to

$$b = S - \frac{1}{3} (\text{tr}_\delta S) \delta, \quad (8.20)$$

so $b - (\text{tr}_\delta b)\delta = S$. The scalar construction gains two Sobolev derivatives. The symmetric divergence construction first gains one derivative, uses a further divergence inverse to correct the skew part, and differentiates that correction once; its net gain is one. This proves (8.19), with support in a fixed larger compact subset and linear dependence on the sources. No quantitative derivative bound for the initial data is being inferred here; the commutator calculation below establishes the needed source norms from O_2/O_1 .

For necessity, integration by parts pairs $\mathcal{L}b$ with $\text{Hess } \phi - (\Delta\phi)\delta$, which vanishes for affine ϕ , and pairs $\mathcal{D}k$ with the symmetric gradient of a Killing field after trace reversal. The latter also vanishes. Thus all and only the stated moments obstruct these inverses with compactly supported output. $\square$

8.7 Rest-end replacement and comparison of all cuts

Proof of Proposition 8.4. Use Lemma 8.8 for a reference end with exactly the original charges. We first join the data to this plane on $R < r < 4R$, controlling the full pointwise constraint deficit.

To estimate the error from joining the two data sets, fix $1 < p < \min(2, q)$, write $x = Rz$, and on $1 < |z| < 4$ use the scaled component fields

$$g(Rz), \quad k(z) = RK(Rz),$$

and their reference counterparts. This corresponds to rescaling lengths by R^{-1} , so both constraint densities scale by R^2 . The metric deviations and tensors have size $O(R^{-p})$ in C^2 and C^1 , respectively. Expanding the constraint map about $(\delta, 0)$ gives its linear part $(\mathcal{L}, \mathcal{D})$ from (8.17), with remainder bounded in C^0 by CR^{-2p} . This follows directly from the scalar curvature formula: the nonlinear metric terms are bounded by $C(|b||D^2b| + |Db|^2)$, where $b = g(Rz) - \delta$ here; the other scalar terms are quadratic in k . The momentum remainder is bounded by $C(|b||Dk| + |Db||k|)$.

Choose a fixed smooth radial ψ , equal to one near $|z| = 1$ and zero near $|z| = 4$, with $0 \leq \psi \leq 1$. Blend the metric components and tensors by ψ . If b is the difference of the two scaled metric perturbations and k the difference of the two scaled tensors, the errors of the linear constraints relative to the blended sources are $f_R = [\mathcal{L}, \psi]b$ and $F_R = [\mathcal{D}, \psi]k$. For $T_{ij} = b_{ij} - (\text{tr}_\delta b)\delta_{ij}$,

$$[\mathcal{L}, \psi]b = (\partial_i \partial_j \psi)T_{ij} + 2(\partial_i \psi)\partial_j T_{ij}, \quad (8.21)$$

$$([\mathcal{D}, \psi]k)_i = (\partial_j \psi)(k_{ij} - (\text{tr}_\delta k)\delta_{ij}). \quad (8.22)$$

These expressions are supported in a fixed compact subannulus, and their C^1 norms are $O(R^{-p})$. In particular, the first bound uses only two metric derivatives, and the second only one tensor derivative.

To control the error remaining after correction, we prove the sharper bound $o(R^{-2})$ for every moment in (8.18). Work momentarily in the physical coordinates, with

$$h = g - g_*, \quad T = K - (\text{tr}_\delta K)\delta - K_* + (\text{tr}_\delta K_*)\delta, \quad f = \mathcal{L}h, \quad F_i = \partial_j T_{ij}.$$

Both f and F belong to L^1 on the end. Indeed, the difference between the physical constraints and their Euclidean linear parts is $O(r^{-2-2\min(q,2)})$, which is integrable in dimension four. The physical densities are integrable by hypothesis and by reference vacuum, and Euclidean and physical measures and norms are uniformly comparable there.

An elementary consequence of integrability is

$$\int_{r_0 < r < 4R} r(|f| + |F|) \, dx = o(R). \quad (8.23)$$

To see this, split at a fixed radius L . The integral up to L , divided by R , tends to zero. The remaining integral divided by R is at most four times the L^1 tail beyond L , which can be made arbitrarily small. A finite first moment is not required.

For an affine function ϕ , define the vector field

$$Q_{\phi,i} = \phi(\partial_j h_{ij} - \partial_i h_{jj}) - (\partial_j \phi) h_{ij} + (\partial_i \phi) h_{jj}.$$

Its divergence is ϕf . For a Killing field Y , symmetry of T gives $\partial_j(Y^i T_{ij}) = Y^i F_i$. Let $\mathfrak{F}_\phi(r)$ and $\mathfrak{G}_Y(r)$ be the corresponding outward Euclidean sphere fluxes. For constant tests, matching the ADM charges gives

$$\mathfrak{F}_1(r) = o(1), \quad \mathfrak{G}_{e_i}(r) = o(1).$$

In the momentum flux the replacement of $\text{tr}_g K g$ by $(\text{tr}_\delta K) \delta$ costs at most $O(r^{2-2\min(q,2)}) = o(1)$. For homogeneous linear scalar tests and rotational fields, integration from a fixed sphere and (8.23) instead give

$$\mathfrak{F}_\phi(r) = o(r), \quad \mathfrak{G}_Y(r) = o(r).$$

Put $\psi_R(x) = \psi(x/R)$, and let $A_R = \{R < r < 4R\}$. Integration by parts, with $\psi_R = 1$ near the inner sphere and zero near the outer sphere, gives the exact formulas

$$\int_{A_R} \phi[\mathcal{L}, \psi_R] h \, dx = -\mathfrak{F}_\phi(R) - \int_{A_R} \psi_R \phi f \, dx, \quad (8.24)$$

$$\int_{A_R} Y^i ([\mathcal{D}, \psi_R](K - K_*)_i) \, dx = -\mathfrak{G}_Y(R) - \int_{A_R} \psi_R Y^i F_i \, dx. \quad (8.25)$$

In scaled coordinates, constant moments have the factor R^{-2} : the source scales by R^2 and volume by R^{-4} . A linear test in z contributes the additional factor R^{-1} . Thus (8.24)–(8.25) and the preceding estimates show

$$\int f_R \phi \, dz = o(R^{-2}), \quad \int (F_R)_i Y^i \, dz = o(R^{-2}) \quad (8.26)$$

for each element of a fixed basis of the affine functions or Killing fields.

We remove these small moments before applying the compactly supported inverses. Choose a nonnegative smooth bump ζ , supported in a fixed ball within the annulus and positive on a smaller ball. For a basis of the affine tests, its Gram matrix $\int \zeta \phi_a \phi_b$ is positive definite. For a basis of Killing fields, the matrix $\int \zeta Y_a \cdot Y_b$ is likewise positive definite: a nonzero affine Killing field cannot vanish on an open ball. Subtract the corresponding bump combinations from f_R, F_R to erase their moments. By (8.26), the coefficients, and every fixed smooth norm of these bump corrections, are $o(R^{-2})$.

Apply Lemma 8.9 to the negatives of the moment-free commutators and add the resulting tensors to the blend. With $j = 1$ and $p_1 > 4$, Sobolev embedding and (8.19) bound the metric correction in C^2 and the tensor correction in C^1 by CR^{-p} . Their supports stay in a fixed enlarged compact subannulus. The new metric is positive for sufficiently large R and agrees exactly with the original data inside R and the plane outside $4R$.

Let $(\tilde{g}_R, \tilde{K}_R)$ denote these physical data. Their scaled constraints equal ψ times the original scaled constraints, plus an error $o(R^{-2}) + O(R^{-2p})$; the reference constraints vanish. The change in the momentum norm costs another $O(R^{-2p})$, because the metric change is $O(R^{-p})$ and the scaled momentum density is $O(R^{-p})$. Since $\psi \geq 0$, the original dominant energy condition implies, after undoing the scaling,

$$\tilde{\mu}_R - |\tilde{J}_R|_{\tilde{g}_R} \geq -\eta_R R^{-4}, \quad \eta_R \longrightarrow 0. \quad (8.27)$$

Indeed the remaining physical error is $R^{-2}[o(R^{-2}) + O(R^{-2p})] = o(R^{-4})$, since $p > 1$. The possible deficit is confined to a fixed closed subannulus on the scale R . Replace η_R by a positive majorant tending to zero if necessary.

We next bend the plane to a static slice within the exact reference region. In its spatial coordinates y , put $v = P/E \in \mathbb{R}^4$ and write the plane as $b = v \cdot y$. Choose a smooth switch ψ_1 from one to zero on $[0, 1]$, constant outside that interval, and replace the plane farther out by the graph

$$b = (v \cdot y)\psi_1 \left(L^{-1} \log(|y|/R_b) \right). \quad (8.28)$$

Here R_b is a sufficiently large fixed multiple of R , so the graph initially lies wholly beyond the gluing annulus. For fixed $|v| < 1$, choose the fixed L so large that

$$|\text{grad}_y b| \leq |v|(1 + \|\psi'_1\|_\infty/L) < 1.$$

The Minkowski graph is uniformly spacelike. Since the perturbation (8.16) is $O(r^{-2})$, it remains uniformly spacelike in that metric for large R . It agrees with the plane on an open inner region and with $b = 0$ on an open outer region. These induced data are vacuum everywhere in the bend, so it adds no deficit.

In the fixed x chart, use $y = Ax$ where $A = (I - v \otimes v)^{-1/2}$, with the chosen rotation. The derivatives of (8.28) give $D^2b = O(r^{-1})$, and the induced data, now denoted (g_R^0, K_R^0) , satisfy on all transition regions

$$c\delta \leq g_R^0 \leq C\delta, \quad |\partial g_R^0| + |K_R^0| \leq C/r. \quad (8.29)$$

Here $c, C > 0$ may depend on the fixed boost and switches, but not on large R . Outside a fixed multiple of R , the data are static in the y chart and have tensor zero.

For the repair choose a smooth radius $\mathfrak{r}$, equal to $|x|$ through the gluing annulus and through the bend, and equal to $|y|$ farther out in the static region. It can be chosen with

$$\mathfrak{r} \asymp r, \quad c \leq |\text{d}\mathfrak{r}|_{g_R^0} \leq C, \quad |\Delta_{g_R^0} \mathfrak{r}| \leq C/r. \quad (8.30)$$

Here is an explicit way to make the interpolation. In the static region put $a(n) = |A^{-1}n|$, $n = y/|y|$, and interpolate

$$\log \mathfrak{r} = \log |y| + (1 - \chi(\log(|y|/R_c))) \log a(n).$$

Take R_c beyond the bend and make χ change from zero to one over a sufficiently long fixed logarithmic interval. Its slope can be made so small that the derivative of $\log \mathfrak{r}$ with respect to $\log |y|$ lies between $1/2$ and $3/2$. Angular derivatives are bounded, second spatial derivatives are $O(r^{-1})$, and (8.29) proves (8.30). All transition intervals are contained between fixed multiples of R .

To restore the dominant energy condition, we construct a conformal factor whose contribution to the mass tends to zero. Write $s = \mathfrak{r}/R$. We construct a nonnegative function $F_R(s)$, constant on the inside, and use

$$u_R = 1 + \varepsilon_R R^{-2} F_R(s), \quad \varepsilon_R = C_0 \eta_R.$$

Choose a fixed interval $[a, b_1]$ in the s variable that starts before the possible deficit, contains all transition regions, and ends where the radius is $|y|$ and the data are static. This interval lies in the end for all large R . Put $z_R = -F'_R$ and prescribe

$$z_R(a) = 0, \quad z'_R = C_1 z_R + \omega_1, \quad (8.31)$$

where $\omega_1 \geq 0$ is smooth, vanishes on a neighborhood of a , and is at least one on a neighborhood of the possible deficit interval. Choose it to vanish near b_1 . The solution is nonnegative and its norms on this fixed interval are bounded independently of large R .

Differentiating u_R and using (8.29) and (8.30) gives fixed constants $c', C' > 0$ such that

$$-\Delta_{g_R^0} u_R - |K_R^0| |du_R|_{g_R^0} \geq \varepsilon_R R^{-4} (c' z'_R - C' z_R) \quad \text{on } a \leq s \leq b_1. \quad (8.32)$$

In fact the first positive term is $\varepsilon_R R^{-4} z'_R |d\mathbf{r}|^2$; the other terms are $\varepsilon_R R^{-3} z_R \Delta \mathbf{r}$ and the negative tensor term, both bounded below by $-C \varepsilon_R R^{-4} z_R$ here. Choose $C_1 > C'/c'$. Equation (8.31) then makes (8.32) nonnegative everywhere on this interval and at least $c' \varepsilon_R R^{-4}$ on the deficit region.

On the static region put $c_m = m/2$ and continue z_R so that

$$q_R(s) = s^3 \left(1 + \frac{c_m}{R^2 s^2} \right)^2 z_R(s)$$

is nondecreasing and becomes a positive constant $q_{R,\infty}$. This can be done smoothly with a uniform bound on $q_{R,\infty}$. Indeed, immediately past b_1 , continue (8.31); both its derivative and that of the positive prefactor make q'_R nonnegative for large R . Multiply this nonnegative derivative by a smooth switch that equals one initially and zero after one more fixed interval, and integrate. This preserves all matching derivatives and keeps q_R nondecreasing. The radial Laplacian of the static metric then shows $-\Delta_{g_R^0} u_R \geq 0$ there, since its radial divergence is proportional to $-q'_R$; also $K_R^0 = 0$.

Set $F_R(s) = \int_s^\infty z_R(t) dt$, extending it constantly before a . Its norm is bounded uniformly: the finite intervals have bounded length and data, while on the tail $z_R = O(s^{-3})$. Thus $1 \leq u_R \leq 2$ for large R . Choose the fixed C_0 large enough that $3u_R^{-1} c' \varepsilon_R \geq \eta_R$. Lemma 8.7, (8.27), and (8.32) now prove the dominant energy condition everywhere for

$$g_R = u_R^2 g_R^0, \quad K_R = u_R K_R^0.$$

The repair is constant in a neighborhood of the original boundary, and therefore preserves its expansion sign.

On the ultimate tail the integral for F_R is explicit:

$$u_R = 1 + \frac{a_R}{|y|^2 + c_m}, \quad a_R = \frac{1}{2} \varepsilon_R q_{R,\infty} = O(\varepsilon_R) = o(1).$$

Consequently

$$g_R = \left(1 + \frac{c_m + a_R}{|y|^2} \right)^2 \delta, \quad K_R = 0 \quad \text{on that tail.} \quad (8.33)$$

Its energy is $m + 2a_R = m + o(1)$, with zero momentum, by the direct static instance of Lemma 8.8. The densities vanish on this tail and are smooth on the remaining compact part, so they are integrable. The data are complete with boundary: a fixed compact core is smooth, and the end has a uniform positive metric lower bound for each R .

It remains to compare the areas of all enclosing cuts. The final metric dominates the original g for $r \leq R$, because the intermediate data there are the original ones and $u_R \geq 1$. For $r \geq R$, (8.29) and the static tail give $g_R \geq c\delta$ in the original x chart with a fixed $c > 0$. Choose $0 < \kappa < \min(1, \sqrt{c}/2)$. Let $L_R = R^{1/2}$, and choose a smooth strictly increasing radial map ρ_R

equal to r for $r \leq L_R$, with derivative decreasing from one to κ on $[L_R, 2L_R]$, and derivative κ afterward. It defines a diffeomorphism

$$\Phi_R(x) = \rho_R(|x|) \frac{x}{|x|}$$

on the end, extended by the identity on the core. It is proper because $\rho_R(r) \rightarrow \infty$, preserves the selected end, and fixes the boundary. Its radial and tangential Euclidean dilations are ρ'_R and ρ_R/r . They are at most one everywhere, and for $r \geq R$ are at most $\kappa + O(R^{-1/2})$.

On the nonidentity region with $r \leq R$, both r and $\rho_R(r)$ tend uniformly to infinity; the original metric is therefore $(1 + o(1))\delta$ at both points. The dilation bound and $g_R \geq g$ show $\Phi_R^*g \leq (1 + o(1))g_R$ there. For $r \geq R$, the choice of κ , the same original-end decay, and $g_R \geq c\delta$ give this inequality as well. It is immediate on the identity core. We have thus proved the global tensor comparison

$$\Phi_R^*g \leq (1 + \epsilon_R)g_R, \quad \epsilon_R \longrightarrow 0. \quad (8.34)$$

Let Γ be the full boundary of any admissible outer domain for g_R . Its image under this proper diffeomorphism is the full boundary of an admissible original outer domain, with all coincident frontier retained. Equation (8.34) gives

$$|\Phi_R(\Gamma)|_g \leq (1 + \epsilon_R)^{3/2} |\Gamma|_{g_R}.$$

Taking infima gives

$$A_*(g) \leq (1 + \epsilon_R)^{3/2} A_*(g_R),$$

which proves the last assertion without a compactness assumption on minimizing sequences. $\square$

8.8 Strict trapping and a positive energy margin

The rest-end replacement preserves weak trapping and the dominant energy condition. The construction of Sections 3–5 requires strict inequalities. We now obtain them by a conformal change whose size tends uniformly to zero and whose change in ADM energy can be computed exactly. All constants in this step belong to one fixed rest-end metric. In particular, when that metric depends on the replacement radius R , we keep R fixed throughout this construction.

The Poisson equation with a gradient-dependent source used below is related to Jaracz's conformal deformation to the strict dominant energy condition [29]. The proof here supplies the four-dimensional equation, its nonzero inner Neumann condition, and the charge and enclosing-area estimates needed for the numerical argument.

Proof of Lemma 8.5. Lemma 6.5 applies to this fixed rest exterior. Indeed the exact static tail has $g - \delta = O_d(r^{-2})$ for every d , and K is compactly supported. Choose any $1 < q_* < 2$ to express its O_6/O_5 bounds, set $n = 4$, $k = 3$, and take $\beta = \delta_1$. The condition $0 < \delta_1 < \min(1, q_*)$ is exactly the one required by that lemma. Its smooth nonnegative majorant can be chosen compactly supported: write it as $b_0 \geq |K|_g$. That lemma therefore gives a positive smooth solution of

$$-\Delta_g \phi = b_0 \sqrt{|d\phi|_g^2 + r^{-6} + r^{-4-\delta_1}}, \quad \partial_\nu \phi = -1, \quad \phi \longrightarrow 0. \quad (8.35)$$

The exponent -6 is $-2k$. The all-order clause of the same lemma gives

$$|\partial^j \phi| \leq C_j r^{-2-j} \quad (j \geq 0). \quad (8.36)$$

This application occurs after rest replacement, so it uses no higher derivatives of the original end with the stated two metric derivatives and one tensor derivative.

We now verify the geometric conclusions. Put

$$F = b_0 \sqrt{|\mathrm{d}\phi|^2 + r^{-6}} + r^{-4-\delta_1}.$$

The inner outward normal is $-\nu$, and the finite-flux conclusion of the lemma is

$$L_\phi = \lim_{T \rightarrow \infty} \int_{S_T} \partial_r \phi \, \mathrm{d}A_\delta = -|\partial M|_g - \int_M F \, \mathrm{d}V_g < 0. \quad (8.37)$$

The geometric and Euclidean fluxes differ by $O(T^{-2})$ on the static tail, by (8.36).

Apply Lemma 8.7 with $u = e^{s\phi}$. It gives

$$e^{2s\phi}(\mu_s - |J_s|_{g_s}) \geq 3s(r^{-4-\delta_1} - s|\mathrm{d}\phi|^2).$$

The number $B = \sup_M r^{4+\delta_1} |\mathrm{d}\phi|^2$ is finite: smoothness controls the compact core and the tail is $O(r^{-2+\delta_1})$. Choose $sB \leq 1/2$. This proves the global margin with $c_s = (3s/2)e^{-2s\|\phi\|_\infty} > 0$. On each boundary component,

$$\Theta_{K_s} = e^{-s\phi}(\Theta_K + 3s\partial_\nu \phi) = e^{-s\phi}(\Theta_K - 3s) < 0.$$

Furthermore

$$g \leq g_s \leq e^{2s\|\phi\|_\infty} g. \quad (8.38)$$

Thus completeness with the boundary included is preserved, and the tensor remains compactly supported. On the tail, $R_g = 0$, $K_s = 0$ and

$$R_{g_s} = 6e^{-2s\phi}(sr^{-4-\delta_1} - s^2|\mathrm{d}\phi|^2) = O(r^{-4-\delta_1}).$$

This proves the scalar decay and integrability there; the rest is smooth and compact. All differentiated metric bounds follow from (8.36), and momentum is zero because the tensor vanishes on a distant tail.

The leading metric change is $2s\phi\delta + O_1(r^{-4})$, whose energy flux numerator is $-6s\partial_i \phi + O(r^{-5})$. The latter remainder has vanishing sphere flux. Division by $6\omega_3$ therefore gives

$$E_{g_s} - E_g = -\frac{s}{\omega_3} L_\phi \longrightarrow 0. \quad (8.39)$$

Every enclosing hypersurface has its three-dimensional area element multiplied by $e^{3s\phi}$; the cut class itself is unchanged. Taking infima yields

$$A_*(g) \leq A_*(g_s) \leq e^{3s\|\phi\|_\infty} A_*(g). \quad (8.40)$$

The infimum is finite, since a distant coordinate sphere is admissible. Its convergence as $s \downarrow 0$ follows without an attained minimizer. Together these estimates prove every asserted output at this fixed rest datum. If it is (g_R, K_R) , every constant above may depend on R ; strict conformal deformation is removed with R fixed. $\square$

8.9 The general maximal DEC energy inequality

The maximal special case requires no assumption that a momentum limit exists. The following end calculation supplies that limit from the decay and integrability already imposed on the data.

Lemma 8.10 (Existence of the momentum flux). *On a smooth four-dimensional asymptotically flat end, suppose $g - \delta = O_2(r^{-q})$ and $K = O_1(r^{-1-q})$ for some $q > 1$. If $|J|_g$ is integrable, where $J_i = \nabla^j(K_{ij} - (\text{tr}_g K)g_{ij})$, then each ADM momentum limit in (8.2) exists and is finite.*

Proof. In the given coordinates set

$$\pi_{ij}^g = K_{ij} - (\text{tr}_g K)g_{ij}, \quad \pi_{ij}^\delta = K_{ij} - (\text{tr}_\delta K)\delta_{ij}.$$

The differentiated decay assumptions give

$$\pi^g - \pi^\delta = O_1(r^{-1-2q}), \quad \partial_j \pi_{ij}^\delta - J_i = O(r^{-2-2q}). \quad (8.41)$$

To verify the second bound, expand the covariant divergence as

$$J_i = g^{jk}(\partial_k \pi_{ij}^g - \Gamma_{ki}^a \pi_{aj}^g - \Gamma_{kj}^a \pi_{ia}^g).$$

Replacing g^{jk} by δ^{jk} costs $O(r^{-q})O(r^{-2-q})$; replacing π^g by π^δ costs the derivative of the first bound in (8.41). Each connection term has size $O(r^{-1-q})O(r^{-1-q})$. These are exactly the asserted errors.

Their Euclidean radial volume bound is Cr^{1-2q} , integrable because $q > 1$. The asymptotic metric and Euclidean volume and covector norms are uniformly comparable, so $\partial_j \pi_{ij}^\delta \in L^1$. The Euclidean divergence theorem between two large spheres makes the flux $\int_{S_R} \pi_{ij}^\delta n_\delta^j dA_\delta$ a Cauchy function of R , with finite limit. By the first bound in (8.41), its difference from the flux of π^g is $O(R^{2-2q}) \rightarrow 0$. This is the tensor appearing in (8.2), which proves the lemma. $\square$

Corollary 8.11 (Maximal DEC energy inequality). *Let (X^4, g_X, K_X) be smooth and connected, with X orientable, g_X complete as a metric space with its nonempty compact smooth boundary S_X included, and with exactly one end, which is asymptotically flat and diffeomorphic to the complement of a closed ball in $\mathbb{R}^4$. Let K_X be a smooth symmetric covariant two-tensor. Suppose*

$$\begin{aligned} \text{tr}_{g_X} K_X &= 0, & \mu_X &= \frac{1}{2}(R_{g_X} - |K_X|_{g_X}^2) \geq |\text{div}_{g_X} K_X|_{g_X}, \\ \mu_X, |\text{div}_{g_X} K_X|_{g_X} &\in L^1(X, dV_{g_X}). \end{aligned}$$

Assume for some $q > 1$ that

$$g_X - \delta = O_2(r^{-q}), \quad K_X = O_1(r^{-1-q}),$$

and that the ADM energy E_X in (8.1) is finite. On every component of S_X , require $H + \text{tr}_{S_X} K_X \leq 0$ with the normal into X . For the one-ended enclosing-cut infimum of Definition 8.1, denoted here by $a_{g_X}(S_X)$, one has

$$E_X \geq \frac{1}{2} \left(\frac{a_{g_X}(S_X)}{2\pi^2} \right)^{2/3}.$$

The boundary may have any topology and finitely many components. The momentum limit follows from the stated hypotheses.

Proof. Maximality identifies the momentum density with $J_X = \operatorname{div}_{g_X} K_X$. Lemma 8.10 supplies the finite ADM momentum. All hypotheses of Theorem 8.2 therefore hold for this one-ended exterior. Its invariant-mass inequality in particular gives the displayed energy bound. The proof of Theorem 8.2 uses Theorem 2.3, independently of this corollary. This numerical consequence retains its broad maximal-DEC scope; the separate direct maximal-vacuum construction has its own hypotheses. $\square$

A Derivation of the scalar identity

Proof of Proposition 3.1. For symmetric tensors define the bilinear form and trace reversal

$$\mathbf{B}(B, C) = \langle B, C \rangle_g - (\operatorname{tr}_g B)(\operatorname{tr}_g C), \quad P_B = B - (\operatorname{tr}_g B)g.$$

Put $U = l\sqrt{D}$ and $\bar{g} = g + l^2 df \otimes df$. We derive the curvature formula before completing any squares. On $\mathbb{R}_z \times \mathcal{M}_N$ consider the stationary Lorentz metric

$$\mathbf{G} = -l^2(dz - df)^2 + \bar{g} = -l^2 dz^2 + 2l^2 dz df + g.$$

The constant- z slices have lapse U , shift $Uw = l^2 \nabla f$, future unit normal $\mathbf{n} = U^{-1} \partial_z - w$, and second fundamental form

$$b = -U^{-1} \operatorname{sym} \nabla(Uw) = -H^f - p(dt \otimes w + w \otimes dt). \quad (\text{A.1})$$

This sign follows directly from $\partial_z g = 2Ub + \mathcal{L}_{Uw}g = 0$ and the prescribed future-normal convention. Write $B_0 = \operatorname{tr} b$. The contracted Gauss equation and the mean-curvature evolution equation for the lapse U give

$$\mathbf{R} + 2\operatorname{Ric}(\mathbf{n}, \mathbf{n}) = R_g - \mathbf{B}(b, b), \quad \operatorname{Ric}(\mathbf{n}, \mathbf{n}) = -\mathbf{n}(B_0) - |b|^2 + U^{-1} \Delta_g U.$$

Since B_0 is stationary, $\mathbf{n}(B_0) = -w(B_0)$, and taking the trace in Equation (A.1) gives $\operatorname{div}(Uw) = -UB_0$. Substitution proves

$$\mathbf{R} = R_g + \mathbf{B}(b, b) - 2U^{-1} \operatorname{div}(\nabla U + B_0 Uw). \quad (\text{A.2})$$

In coordinates $z' = z - f$, the same metric is the static warped product $-l^2(dz')^2 + \bar{g}$. Its only mixed connection terms are $\Gamma_{iz'}^{z'} = \partial_i \log l$ and $\Gamma_{z'z'}^i = l \bar{g}^{ij} \partial_j l$; contracting their curvature contributions yields $\mathbf{R} = R_{\bar{g}} - 2l^{-1} \Delta_{\bar{g}} l$. Also $\bar{g}^{-1} = A_d$, $dV_{\bar{g}} = \sqrt{D} dV_g$, and therefore

$$l^{-1} \Delta_{\bar{g}} l = U^{-1} \operatorname{div}((U/l) A_d \nabla l).$$

Differentiating $U = l\sqrt{D}$ and using Equation (3.4) proves the vector identity

$$\nabla U - (U/l) A_d \nabla l = -Ub(w, \cdot).$$

Consequently Equation (A.2) becomes

$$\frac{1}{2} R_{\bar{g}} = \frac{1}{2} R_g + \frac{1}{2} \mathbf{B}(b, b) + U^{-1} \operatorname{div}(UP_b w). \quad (\text{A.3})$$

Set $Q = K - b = L + p(dt \otimes w + w \otimes dt)$. For any symmetric C , the product rule and Equation (A.1) give the useful exact contraction

$$U^{-1} \operatorname{div}(UP_C w) = (\operatorname{div} P_C)(w) - \mathbf{B}(C, b).$$

Expanding $\mathbf{B}(K - b, K - b)$ and applying this formula to $C = K$ transforms Equation (A.3) into

$$\frac{1}{2}R_{\bar{g}} = \mu + J(w) + \frac{1}{2}\mathbf{B}(Q, Q) - U^{-1}\operatorname{div}(UP_Q w). \quad (\text{A.4})$$

For example, the terms involving K on the right cancel to $R_g/2 - \mathbf{B}(b, b)/2 + (\operatorname{div} P_b)(w)$, as they must.

The conformal scalar law in dimension $k + 1$ is

$$\frac{1}{2}e^{2t}R_{\hat{g}} = \frac{1}{2}R_{\bar{g}} - k\Delta_{\bar{g}}t - \frac{k(k-1)}{2}|dt|_{A_d}^2.$$

The volume formula gives

$$\Delta_{\bar{g}}t = U^{-1}\operatorname{div}(UA_d\nabla t) - p|dt|_{A_d}^2.$$

Changing the divergence weight from U to $u = e^{(k-1)t}U$ now yields

$$\begin{aligned} \frac{1}{2}e^{2t}R_{\hat{g}} &= \mu + J(w) + \frac{1}{2}\mathbf{B}(Q, Q) + (k-1)P_Q(w, dt) + ks_p|dt|_{A_d}^2 \\ &\quad - u^{-1}\operatorname{div}(u(P_Q w + kA_d\nabla t)). \end{aligned} \quad (\text{A.5})$$

In particular the last change of weight adds $k(k-1)|dt|_{A_d}^2$ to the preceding coefficient $k[p - (k-1)/2]$.

There are two remaining first-order algebraic identities:

$$\begin{aligned} u(P_Q w + kA_d\nabla t) &= V - uFw, \\ u^{-1}\operatorname{div}(uw) &= \operatorname{tr}_g L - \tau + 2s_p w(t) = F + (1 - \chi)j - \tau + 2s_p a_w x. \end{aligned} \quad (\text{A.6})$$

For the first, $P_Q w + kA_d dt$ has transverse and axial components

$$a_w M + (k + pv)y, \quad -a_w \operatorname{tr} T + kdx,$$

whereas V/u has components $a_w M + (k + pv)y$ and $\chi a_w j + kdx$ by Equation (3.4). Their difference is Fw . For the second, use $\operatorname{div}(Uw) = -U \operatorname{tr} b$ and change the weight. Thus the smooth invariant definition of the claimed quadratic form is

$$\mathcal{T} = \frac{1}{2}\mathbf{B}(Q, Q) + (k-1)P_Q(w, dt) + ks_p|dt|_{A_d}^2 + F(\operatorname{tr}_g L + 2s_p w(t)). \quad (\text{A.7})$$

This proves smoothness and Equation (3.5). To obtain its displayed block form, the blocks of Q are T , $M + pa_w y$, and $j + 2pa_w x$. The transverse terms are

$$|M + pa_w y|^2 + (k-1)a_w(M + pa_w y) \cdot y = |M + s_p a_w y|^2 - \frac{(k-1)^2}{4}v|y|^2.$$

The terms containing $Fa_w x$ cancel, while their remaining axial cross term is $2s_p \chi j a_w x$. Finally $|T|^2 = |T^0|^2 + (F - \chi j)^2/k$ gives Equation (3.6).

For completeness this derivation retains derivatives of p . In the flux of Equation (A.5) the part explicitly containing p is $p\{v\nabla t - w(t)w\}$; its divergence contributes $p'v|y|^2$. Thus the scalar side contains $-p'v|y|^2$. In Equation (3.5) the redistribution of this term can also be checked directly. Holding t, dt, df, H^f, K fixed when taking the partial derivative in p , one has $\chi_p = -\chi v/(k + pv)$. The terms containing p' in $w(F)$ and $u^{-1}\operatorname{div} V$ are respectively

$$p'\chi_p a_w x j, \quad p'(v|y|^2 + \chi_p a_w x j).$$

Their difference is exactly $-p'v|y|^2$. No derivative of the cutoff has been discarded. $\square$

B Proofs of the local scalar estimates

The two estimates stated in Section 4 are proved here in their full interior and constant-Dirichlet forms. Their application to the coupled equations remains in Proposition 4.6.

B.1 The gradient estimate

Proof of Lemma 4.4. We use only the scalar linear De Giorgi–Nash and weak Harnack estimates, the Krylov–Safonov interior estimate for strong nondivergence solutions, and the Euclidean Hessian estimates of Calderón and Zygmund; see [22]. The two structural arguments needed to apply these results are proved below. All coordinate norms of matrices use the Frobenius norm unless indicated otherwise.

1. *A Hessian estimate with a strict absorption margin.* Normalize the metric at the center of a chart. The coordinate principal matrix is

$$a^{ij} = g^{ij} - (1 - \chi)e^i e^j.$$

If $g = I$, then $|I - a| = 1 - \chi \leq 1 - c$. Consequently, on a sufficiently small chart, depending only on c and the metric modulus,

$$|I - a| \leq 1 - \frac{c}{2}. \quad (\text{B.1})$$

This remains true for the reflected metrics used below, since they are continuous and uniformly Lipschitz. In particular this assertion places no condition on the oscillation of e or χ .

Let $\mathcal{R} = D^2\Delta^{-1}$ on $\mathbb{R}^n$, with its matrix-valued range given the Frobenius norm. Plancherel's identity gives $\|\mathcal{R}\|_{L^2 \rightarrow L^2} = 1$. Its L^4 norm is finite by the Calderón–Zygmund theorem. Interpolation therefore supplies $s_0 > 2$, depending only on n, c , such that

$$\|\mathcal{R}\|_{L^s \rightarrow L^s} (1 - c/2) < 1, \quad 2 \leq s \leq s_0.$$

For a compactly supported v the identity $\Delta v = a : D^2 v + (I - a) : D^2 v$ now gives, by absorption,

$$\|D^2 v\|_{L^s} \leq C \|a : D^2 v\|_{L^s}, \quad s = 2, s_0.$$

Cutoffs, first-derivative interpolation and absorption on nested balls give the local version

$$\|D^2 v\|_{L^s(B_{r/2})} \leq C (\|a : D^2 v\|_{L^s(B_r)} + r^{-2} \|v\|_{L^s(B_r)}), \quad s = 2, s_0. \quad (\text{B.2})$$

For completeness, let ψ be a smooth cutoff for the nested balls. Applying the compact-support estimate to ψv first produces an additional $(t - \rho)^{-1} \|Dv\|_{L^s(B_t)}$ on $B_\rho \Subset B_t$. The interpolation estimate on a slightly larger ball bounds this by $\varepsilon \|D^2 v\|_{L^s}$ plus $C\varepsilon^{-1}(t - \rho)^{-2} \|v\|_{L^s}$, with the scale factors absorbed into ε . Choose the coefficient of the first term smaller than the square of the geometric ratio of the successive gaps, and sum over increasing nested radii. The resulting geometric series is exactly Equation (B.2). These estimates initially apply to the individually finite Sobolev norms of the functions under consideration and then, by approximation, to $W^{2,s}$ functions.

2. *The exact flux and the constant-value reflection.* Write $P = \nabla_g u$, $H = \text{Hess}_g u$, and $q = |P|_g^2/2$. The equation implies $\Delta_g u = (1 - \chi)H(e, e) + G$. Since $\nabla_g q = H^\sharp P$, it gives the pointwise identity

$$A\nabla_g q - GP = (H^\sharp - (\Delta_g u) \text{id})P =: \mathcal{F}. \quad (\text{B.3})$$

Both sides vanish when $P = 0$, so no choice of axis at those points affects this identity. For smooth u , commuting third derivatives gives

$$\operatorname{div}_g \mathcal{F} = |H|_g^2 - (\Delta_g u)^2 + \operatorname{Ric}_g(P, P). \quad (\text{B.4})$$

For C^2 functions the same formula holds distributionally: approximate u in C^2 on compact subsets of each original smooth side. The flux and the right side converge uniformly, so integration against a smooth compactly supported test function passes to the limit. This argument establishes only the geometric divergence identity; the algebraic Equation (B.3) is applied to the original solution, and no approximant is required to solve the equation. Neither identity differentiates χ .

Put $\gamma = 1 - (1 - c)^2 > 0$. Young's inequality, with the gap $\gamma/2$, gives

$$((1 - \chi)H(e, e) + G)^2 \leq (1 - \gamma/2)|H|_g^2 + C_c G^2.$$

Thus, when $|P|_g \leq 1$ and $s = 1 - |P|_g^2 \geq 0$,

$$\operatorname{div}_g(A\nabla_g s + 2GP) \leq -\gamma|H|_g^2 + C_c G^2 + 2|\operatorname{Ric}_g|_g. \quad (\text{B.5})$$

We record precisely how this inequality is used at the face. Subtract the constant boundary value and take Gaussian coordinates $g = dt^2 + h_{ab}(y, t) dy^a dy^b$ on $t \geq 0$. Extend u oddly and h evenly across $t = 0$, and extend χ evenly and G oddly. To check the reflected equation, let $R = \operatorname{diag}(1, \dots, 1, -1)$ in these coordinates. On the lower side, evaluated at the reflected upper point, the gradient and Hessian transform as $P^e = -RP$ and $H^e = -RHR$. Even extension of χ therefore gives $A^e = RAR$, so $A^e : H^e = -A : H = -G = G^e$. This verifies the equation on both open sides for every measurable $\chi \in [c, 1]$. Tangential derivatives of u vanish on the face, while the normal derivative agrees from both sides. Therefore u is C^1 after reflection and belongs locally to $W^{2,s}$ for every finite s individually. The metric is Lipschitz, smooth on either side, and s is a continuous $W^{1,2}$ function.

Let $J = \sqrt{\det g}$ and define the face mean curvature using the normal $+\partial_t$ by

$$\kappa(y) = \frac{1}{2} h^{ab}(y, 0) \partial_t h_{ab}(y, 0+), \quad p(y) = u_t(y, 0).$$

On the upper side, $H_{ab} = \frac{1}{2}(\partial_t h_{ab})p$ and hence $\mathcal{F}_+^t = -\kappa p^2$. On the lower side the signs of these tangential Hessians reverse and $\mathcal{F}_-^t = \kappa p^2$. Since J is continuous across the face, the density-valued flux $J(A\nabla_g s + 2GP) = -2J\mathcal{F}$ has normal jump

$$j(y) = 4J(y, 0)\kappa(y)p(y)^2.$$

Adding the coordinate vector

$$B_{\text{face}}(y, t) = -j(y)\mathbf{1}_{\{t>0\}}\partial_t \quad (\text{B.6})$$

cancels that interface measure exactly. Indeed its coordinate divergence is $-j(y)\delta_{\{t=0\}}$; no derivative in y occurs, because the vector has only a t component. Its L^∞ norm is at most $C\|\kappa\|_\infty$ under the gradient normalization. All these statements are local, so the coefficient $j(y)$ only needs to be extended constantly in the normal direction over the chart.

On a normalized unit ball let the metric oscillation, Christoffel symbols, smooth-side curvature, face second fundamental form if present, and forcing be at most ε . The reflected and interior cases then both give the distributional inequality

$$\operatorname{div}(a_1 Ds + B_1) \leq -\gamma_1 |H|_g^2 + C\varepsilon, \quad a_1^{ij} = J a^{ij}, \quad \|B_1\|_\infty \leq C\varepsilon, \quad (\text{B.7})$$

where $B_1^i = 2JGP^i + B_{\text{face}}^i$ and $\gamma_1 > 0$ is fixed. The matrix a_1 is symmetric and uniformly elliptic with fixed constants. The face can be absent or can be a plane at any position in the ball.

3. *Gradient-norm drop or approximation by a nonzero affine function.* Fix $r_* \in (0, 1/32)$. For every $b_* > 0$ there are $k_* \in (r_*, 1)$ and $\varepsilon_* > 0$ with the following property. Suppose that $|\nabla_g u|_g \leq 1$ on B_1 , the normalized geometric and forcing errors just listed are at most ε_* , and $u(0) = 0$. Then either

$$\sup_{B_{r_*}} |\nabla_g u|_g \leq k_*, \quad (\text{B.8})$$

or there is an affine L such that

$$\frac{1}{2} \leq |DL| \leq 2, \quad \sup_{B_{r_*}} |u - L| \leq b_* r_*. \quad (\text{B.9})$$

Here is the compactness argument with its energy step. Otherwise choose a sequence with errors tending to zero, no approximation in Equation (B.9), and gradient supremum tending to one on B_{r_*} . Its functions are uniformly Lipschitz and vanish at zero. Equation (B.2), applied to

$$a : D^2 u = G + a^{ij} \Gamma_{ij}^\ell u_\ell,$$

gives uniform local L^2 bounds for $D^2 u$, hence for H and Ds . The weak Harnack inequality applies to the nonnegative weak supersolution s in Equation (B.7) after discarding the negative Hessian term. It gives, for some $p_0 > 0$,

$$\left(\frac{1}{|B_{1/3}|} \int_{B_{1/3}} s^{p_0} \right)^{1/p_0} \leq C \left(\inf_{B_{1/3}} s + \|B_1\|_\infty + \varepsilon \right) \rightarrow 0.$$

The infimum tends to zero because $B_{r_*} \subset B_{1/3}$ and the gradient supremum there tends to one. Thus $s \rightarrow 0$ in measure locally.

To recover the Hessian energy, take a compactly supported cutoff ψ in $B_{1/3}$ and test the same inequality with $\psi^2(b-s)_+$, where $0 < b < 1$ is fixed. Its weak form is

$$\int (a_1 Ds + B_1) \cdot D\varphi \geq \gamma_1 \int |H|_g^2 \varphi - C\varepsilon \int \varphi, \quad \varphi \geq 0.$$

Dropping the nonnegative Hessian term, expanding the test derivative and using Young's inequality proves

$$\int_{\{s < b\}} \psi^2 |Ds|^2 \leq C_\psi b^2 + C_\psi \|B_1\|_\infty^2 + C_\psi b \|B_1\|_\infty + C_\psi \varepsilon b. \quad (\text{B.10})$$

On a fixed compact subset, the integral of $|Ds|$ over $\{s < b\}$ has limsup at most Cb . Its integral over $\{s \geq b\}$ tends to zero by the uniform L^2 bound for Ds and convergence in measure of s . Letting $b \downarrow 0$ yields $Ds \rightarrow 0$ locally in L^1 . Testing Equation (B.7) with a fixed nonnegative smooth cutoff now yields $H \rightarrow 0$ locally in L^2 . Since $\Gamma \rightarrow 0$ and Du is bounded, also $D^2 u \rightarrow 0$ locally in L^2 .

Uniform Lipschitz compactness produces a locally uniform limit. Its distributional Hessian vanishes, so it is affine. Poincaré's inequality and $D^2 u \rightarrow 0$ give convergence of the gradients to its slope in local L^2 . Since $s \rightarrow 0$ in measure and $g \rightarrow I$, that slope has norm one. This contradicts the failure of Equation (B.9) and proves the dichotomy. Increasing k_* if necessary ensures $r_* < k_* < 1$.

Repeated gradient-norm drops will give decay directly. To handle the other alternative, we next improve approximation by a nonzero affine function. Its fixed slope determines the direction to freeze; the axial eigenvalue remains measurable throughout this argument.

4. *The frozen-axis linear equation.* Let e_0 be a fixed Euclidean unit vector. We first prove the quantitative statement

$$\|Y_0\|_{C^{1,\alpha_1}(B_{1/2})} \leq C\|Y_0\|_{W^{2,2}(B_1)} \quad \text{if} \quad \Delta_{e_0^\perp} Y_0 + \chi \partial_{e_0}^2 Y_0 = 0 \text{ a.e. in } B_1, \quad (\text{B.11})$$

for $Y_0 \in W^{2,2}$ and arbitrary measurable $c \leq \chi \leq 1$. Rotate so that $e_0 = \partial_n$ and put $w = \partial_n Y_0 \in W^{1,2}$. Differentiating the displayed equation only in the sense of distributions gives exactly

$$\sum_{a < n} \partial_a^2 w + \partial_n(\chi \partial_n w) = 0, \quad \text{or} \quad \operatorname{div}((I - e_0 \otimes e_0 + \chi e_0 \otimes e_0) Dw) = 0. \quad (\text{B.12})$$

This is a scalar divergence equation for a $W^{1,2}$ weak solution with a measurable uniformly elliptic matrix. The De Giorgi–Nash estimate gives $w \in C^{\alpha_1}$ on smaller balls, for a fixed $0 < \alpha_1 < 1$, quantitatively in its L^2 norm. Testing with $\psi^2(w - w(x))$, where ψ is a cutoff on the ball, and using that Hölder bound gives, uniformly for balls in a smaller fixed ball,

$$\int_{B_r(x)} |Dw|^2 \leq Cr^{n-2+2\alpha_1}. \quad (\text{B.13})$$

The ordinary Laplacian of Y_0 is $f = (1 - \chi)\partial_n w$. By Cauchy–Schwarz,

$$\int_{B_r(x)} |f| \leq Cr^{n-1+\alpha_1}. \quad (\text{B.14})$$

Here is why this controls every component of DY_0 . Cut f off on an intermediate ball and form its Newton potential V . For two points at distance r in a smaller ball, the part of DV within distance $2r$ of either point is bounded by

$$C \sum_{j \geq 0} (2^{-j}r)^{1-n} (2^{-j}r)^{n-1+\alpha_1} \leq Cr^{\alpha_1}.$$

For distances $R \geq 2r$, the difference of the two gradient kernels is at most CrR^{-n} . Summing over dyadic annuli and using Equation (B.14) gives

$$Cr \sum_{r \leq R \leq 1} R^{\alpha_1-1} \leq Cr^{\alpha_1}.$$

The farther part is bounded by the total L^1 norm of f and contributes $O(r)$. The same estimates show that V, DV are bounded. The difference $Y_0 - V$ is harmonic on the smaller ball, so interior harmonic estimates control its gradient and its gradient Hölder norm by its L^2 norm. Together with the initial $W^{2,2}$ bound this proves Equation (B.11). Taking slightly nested balls in this argument yields the stated $B_{1/2}$ estimate.

5. *Improvement near a nonzero affine function.* Fix a universal gradient bound, say $|\nabla_g u|_g \leq 4$. There are $\alpha_2 \in (0, \alpha_1)$, $\lambda \in (0, 1/8)$ and small $b_0, \delta > 0$, depending only on n, c , with the following property. If L is affine and

$$\frac{1}{4} \leq |DL| \leq 4, \quad \|u - L\|_{L^\infty(B_1)} \leq b, \quad 0 < b \leq b_0, \quad \|g - I\|_\infty + \|\Gamma\|_\infty + \|G\|_\infty \leq b\delta, \quad (\text{B.15})$$

then there is an affine L' such that

$$\sup_{B_\lambda} |u - L'| \leq b\lambda^{1+\alpha_2}, \quad |DL' - DL| \leq Cb. \quad (\text{B.16})$$

The metrics may again be smooth or reflected.

Set $Y = (u - L)/b$. Its coordinate equation is

$$a : D^2Y = b^{-1}(G + a^{ij}\Gamma_{ij}^\ell u_\ell),$$

whose right side has absolute value at most $C\delta$. Equation (B.2) and interpolation give

$$\|Y\|_{W^{2,s_0}(B_{7/8})} \leq C. \quad (\text{B.17})$$

Put $e_0 = DL/|DL|$ and $a_0 = I - (1 - \chi)e_0 \otimes e_0$. Since $Du = DL + bDY$ and $|DL| \geq 1/4$, the direction projections obey

$$\|a - a_0\|_{L^{s_0}(B_{7/8})} \leq Cb, \quad \|a - a_0\|_\infty \leq C.$$

Indeed the map taking a nonzero vector to its unit-direction projection has the estimate $|\Pi(v) - \Pi(q)| \leq C \min(1, |v - q|/|q|)$ when $q \neq 0$. At $v = 0$ any assigned unit projection satisfies the same estimate after enlarging C . The change from Du to its metric gradient contributes $O(\|g - I\|_\infty)$. Hölder's inequality, with $1/2 = 1/s_0 + 1/p$ and interpolation of the bounded matrix difference in L^p , therefore gives

$$R := a_0 : D^2Y, \quad \|R\|_{L^2(B_{3/4})} \leq C(\delta + b^\sigma), \quad \sigma = \min\{1, (s_0 - 2)/2\} > 0. \quad (\text{B.18})$$

We remove this residual in $W^{2,2}$, without any Sobolev embedding into C^0 . On the convex ball $\mathcal{B} = B_{3/4}$, integration by parts for a zero-trace function gives

$$\int_{\mathcal{B}} |D^2v|^2 \leq \int_{\mathcal{B}} |\Delta v|^2.$$

The omitted boundary term is the nonnegative integral of the sphere's mean curvature times $(\partial_\nu v)^2$; the identity extends by density to $W^{2,2} \cap W_0^{1,2}$. Thus the Dirichlet solution operator Δ_D^{-1} satisfies $\|D^2\Delta_D^{-1}f\|_2 \leq \|f\|_2$. The map

$$v \mapsto \Delta_D^{-1}(R + (1 - \chi)\partial_{e_0}^2 v)$$

is a contraction, with factor at most $1 - c$, on $W^{2,2}(\mathcal{B}) \cap W_0^{1,2}(\mathcal{B})$ equipped with the Hessian norm. Its fixed point satisfies

$$a_0 : D^2v = R, \quad \|v\|_{W^{2,2}(\mathcal{B})} \leq C\|R\|_2.$$

Consequently $Y_0 = Y - v$ satisfies the homogeneous frozen equation and has bounded $W^{2,2}(\mathcal{B})$ norm. By Equation (B.11), $\|Y_0\|_{C^{1,\alpha_1}(B_{3/8})} \leq C$.

Independently, Y has a uniform C^θ bound on $B_{1/2}$ by the Krylov–Safonov estimate applied to its unfrozen equation: the matrix a is measurable and uniformly elliptic, its source is bounded, and $\|Y\|_\infty \leq 1$. The function Y is individually $W^{2,n}$ (also across the face), so the strong-solution version of that theorem applies. Hence $v = Y - Y_0$ has uniformly bounded C^θ norm on $B_{3/8}$. Its small L^2 norm now implies

$$\|v\|_{L^\infty(B_{1/3})} \leq C\|v\|_2^{2\theta/(n+2\theta)} \leq C(\delta + b^\sigma)^{2\theta/(n+2\theta)}. \quad (\text{B.19})$$

To see the first inequality for small $\|v\|_2$, a value $|v(x)| = m$ forces $|v| \geq m/2$ on a ball of radius $(m/(2C))^{1/\theta}$; integrating its square gives the stated exponent.

Choose $\alpha_2 < \alpha_1$, then λ so small that $C\lambda^{1+\alpha_1} \leq \frac{1}{2}\lambda^{1+\alpha_2}$, and finally δ, b_0 so small that the right side of Equation (B.19) is at most $\frac{1}{2}\lambda^{1+\alpha_2}$. The affine function

$$L' = L + b(Y_0(0) + DY_0(0) \cdot x)$$

then satisfies Equation (B.16).

6. *Iteration, including its transition between the two alternatives.* Decrease b_0 further so that $Cb_0/(1 - \lambda^{\alpha_2}) \leq 1/8$, where C is the slope constant in Equation (B.16). Apply the dichotomy with $b_* = b_0$. At each center, first normalize g there and choose a fixed sufficiently small radius R_0 . Divide heights by KR_0 , where $K \geq \max(1, \|\nabla_g u\|_\infty)$, and subtract the center value. On the rescaled unit ball the gradient is at most one. The forcing is multiplied by R_0/K , the connection and face second form by R_0 , and smooth-side curvature by R_0^2 . Metric oscillation from its center value is $O(R_0)$. Choose R_0 uniformly so that these errors satisfy both the dichotomy threshold and $b_0\delta$.

Denote this normalized function by u_0 . After j successive norm drops, the normalized function is

$$u_j(x) = \frac{u_0(r_*^j x)}{r_*^j k_*^j}.$$

It again has gradient at most one. Its forcing has acquired the factor $(r_*/k_*)^j$, and its metric, connection and face errors have acquired at least the factor r_*^j . All required smallness conditions persist because $r_* < k_*$. If this branch continues forever, then

$$\sup_{B_{r_*^j}^j} |u_0 - u_0(0)| \leq Cr_*^j k_*^j = C(r_*^j)^{1+\alpha_d}, \quad \alpha_d = \frac{\log k_*}{\log r_*} \in (0, 1).$$

If the affine alternative first occurs after j norm drops, rescale its ball once by r_* , retaining its current gradient normalization. The resulting function U has a unit-ball affine error at most b_0 and an initial slope in $[1/2, 2]$. Iterate the affine improvement using

$$U_i(x) = \frac{U(\lambda^i x)}{\lambda^i}, \quad b_i = b_0 \lambda^{i\alpha_2}.$$

Only constants are subtracted from the nonlinear unknown if a renormalization of its value is needed; the affine functions are used solely as comparison functions. In particular the equation whose distinguished direction is the gradient remains valid at every step. The relative error threshold remains valid: geometric and forcing errors acquire a factor λ^i , and

$$\lambda^i b_0 \delta \leq b_i \delta \quad \text{because} \quad \alpha_2 < 1.$$

The gradient remains bounded by its entry normalization, and the affine slopes change by at most Cb_i . Their total change is at most $1/8$, so all slopes remain within $[1/4, 4]$, as required by Equation (B.15).

Take $0 < \alpha \leq \min(\alpha_d, \alpha_2)$. Before affine entry the constant approximations above have error $Cr^{1+\alpha}$. At entry the original radius is $r_e = r_*^{j+1}$ and the gradient normalization is $a_e = k_*^j$. At the subsequent radii $r = r_e \lambda^i$, the affine errors in the original u_0 are bounded by

$$Ca_e r_e \lambda^{i(1+\alpha_2)} = Cr^{1+\alpha} (a_e r_e^{-\alpha}) \lambda^{i(\alpha_2-\alpha)} \leq Cr^{1+\alpha},$$

because $a_e r_e^{-\alpha} = k_*^{-1} r_e^{\alpha_d - \alpha} \leq k_*^{-1}$. Thus the constants remain uniform across every possible entry scale. Intermediate radii are handled by restricting the approximation from the next larger radius, whose ratio is one of the two fixed numbers r_*^{-1} and λ^{-1} .

We have proved that at every center and every sufficiently small radius there is an affine approximation with uniform error $Cr^{1+\alpha}$. The elementary affine Campanato criterion gives the gradient estimate: comparison of the affine functions on concentric radii $r, r/2$ bounds their slope difference by Cr^α ; summation identifies the limiting slope with Du at the center. Comparing the approximations at two centers in a common ball of radius comparable to their distance then bounds the difference of these limiting slopes by that distance to the power α . This proves the first estimate in Equation (4.14), also on the reflected face, with constants uniform on the smaller original chart.

Finally let $L_x(y) = u(x) + Du(x) \cdot (y - x)$ in fixed coordinates. The gradient estimate gives $\|u - L_x\|_{L^\infty(B_{2r}(x))} \leq Cr^{1+\alpha}$. Since

$$a : D^2(u - L_x) = G + a^{ij} \Gamma_{ij}^\ell u_\ell,$$

Equation (B.2) with $s = 2$ yields

$$\|D^2 u\|_{L^2(B_r(x))} \leq C(r^{n/2} + r^{n/2-1+\alpha}).$$

The bounded connection term converting $D^2 u$ to $\text{Hess}_g u$ has L^2 norm $O(r^{n/2})$. Squaring the estimate and using $\alpha < 1$ proves the Hessian estimate in Equation (4.14). Restricting the reflected estimate to the original half-ball proves the face version. $\square$

B.2 The estimate for quadratic gradient growth

Proof of Lemma 4.5. If a Dirichlet face is present, first subtract its constant boundary value. This changes neither the equation nor its source bounds. The reflection creates no measure on that face. Flatten it as $\{x_n = 0\}$ and write $J = \text{diag}(1, \dots, 1, -1)$. For $x_n < 0$ extend the four quantities by

$$Z^e(x) = -Z(Jx), \quad A^e(x) = JA(Jx)J, \quad B^e(x) = -JB(Jx), \quad S^e(x) = -S(Jx). \quad (\text{B.20})$$

The zero trace makes Z^e a $W^{1,2}$ function, and ellipticity is preserved. To verify the equation across the face, test the original equation on the upper half with $\varphi(x) - \varphi(Jx)$, which has zero trace. Changing variables in the reflected half gives precisely Equation (4.15) for the extended quantities. Equivalently, the normal component of the full flux $ADZ + B$ agrees in the weak flux sense on the two sides. Thus there is no boundary flux jump to estimate. Reflect the nonnegative density $1 + F_0$ evenly. The ball-mass bound changes by at most a fixed multiplicative factor. The reflected coefficients need not be smooth.

Exponential absorption and potentials. Use $\nu = (1 + F_0) dx$, with the even extension just described where needed. We give the two-sign exponential argument for Equation (4.15). Put $Q_\varepsilon = e^{\varepsilon LZ}$ for $\varepsilon \in \{-1, 1\}$, and let $\mathcal{L} = -\partial_i(A^{ij}\partial_j)$. The weak chain rule gives

$$\mathcal{L}Q_\varepsilon = \partial_i(\varepsilon LQ_\varepsilon B^i) - L^2 Q_\varepsilon B \cdot DZ + \varepsilon LQ_\varepsilon S - L^2 Q_\varepsilon ADZ \cdot DZ. \quad (\text{B.21})$$

Indeed it follows directly by testing the equation for Z with $\varepsilon LQ_\varepsilon \varphi$. Young's inequality bounds the second term on the right by $\frac{1}{2}\lambda L^2 Q_\varepsilon |DZ|^2 + CL^2 Q_\varepsilon$. Taking $L \geq 2C/\lambda$ absorbs also the quadratic part of $\varepsilon LQ_\varepsilon S$. Since Z is bounded and L is now fixed, we obtain, for both signs,

$$\mathcal{L}Q_\varepsilon \leq \text{div } B_\varepsilon + C\nu, \quad B_\varepsilon = \varepsilon LQ_\varepsilon B, \quad \|B_\varepsilon\|_\infty \leq C. \quad (\text{B.22})$$

This uses no derivative of B , A , or the density of ν .

On a ball B_r in the resulting full patch, let v_ε solve

$$\mathcal{L}v_\varepsilon = \operatorname{div} B_\varepsilon + C\nu, \quad v_\varepsilon|_{\partial B_r} = 0. \quad (\text{B.23})$$

The density of ν is bounded for each individual classical solution, so this solve and the following tests can first be made in $W^{1,2}$. Only the uniform mass bound in Equation (4.16) enters the estimate. The part with right side $\operatorname{div} B_\varepsilon$ has absolute value at most $C\|B_\varepsilon\|_\infty$. This is the scaled bounded-divergence-source estimate: testing by positive and negative level truncations gives $\int |D(v - k)_+|^2 \leq C\|B_\varepsilon\|_\infty^2 |\{v > k\}|$, and Sobolev level iteration on the unit ball proves the asserted bound after rescaling.

For the measure part, the scalar Dirichlet Green function of a uniformly elliptic divergence operator in dimension $n \geq 3$ satisfies $0 \leq G_r(x, y) \leq C|x - y|^{2-n}$. For bounded measurable symmetric coefficients, extend A uniformly elliptically to $\mathbb{R}^n$ and let Γ_A denote the positive whole-space Green function for $-\operatorname{div}(AD)$. Equation (7.9) of [31, p. 67] gives $\Gamma_A(x, y) \leq C|x - y|^{2-n}$. For a fixed pole y , the difference $\Gamma_A(\cdot, y) - G_r(\cdot, y)$ has cancelling point sources, solves the homogeneous equation in the ball, and has nonnegative boundary trace. The maximum principle therefore gives $0 \leq G_r(x, y) \leq \Gamma_A(x, y)$. The bound uses only dimension and ellipticity, so it also applies to the reflected matrix in Equation (B.20). Dividing B_r into annuli about any $x \in B_r$ gives

$$\int_{B_r} G_r(x, y) d\nu(y) \leq C \sum_{j \geq 0} (2^{-j} 2r)^{2-n} \nu(B_{2^{-j} 2r}(x)) \leq C \sum_{j \geq 0} (2^{-j} 2r)^a \leq Cr^a.$$

We work sufficiently far inside the coordinate patch that the mass bound applies to the balls in this sum. Consequently

$$\|v_\varepsilon\|_{L^\infty(B_r)} \leq E_r, \quad E_r = C(r + r^a). \quad (\text{B.24})$$

Oscillation decay. Here is the oscillation reduction, including the role of the two signs. Let $m = \inf_{B_r} Z$, $M = \sup_{B_r} Z$, and $\omega = M - m$. By Equations (B.22) and (B.23), the functions

$$W_+ = e^{LM} - Q_+ + v_+ + E_r, \quad W_- = e^{-Lm} - Q_- + v_- + E_r$$

are nonnegative weak supersolutions of $\mathcal{L}W \geq 0$ on B_r . At least one of the sets

$$\{Z \leq (m + M)/2\} \cap B_{r/2}, \quad \{Z \geq (m + M)/2\} \cap B_{r/2}$$

has at least half the measure of $B_{r/2}$. On the first set $e^{LM} - Q_+ \geq c\omega$, and on the second $e^{-Lm} - Q_- \geq c\omega$, because the exponential and its inverse are uniformly Lipschitz on the fixed bounded range of Z . For the corresponding sign the weak Harnack inequality, with its fixed exponent $q > 0$, yields

$$c\omega \leq C \left(\frac{1}{|B_{r/2}|} \int_{B_{r/2}} W_\varepsilon^q \right)^{1/q} \leq C \inf_{B_{r/4}} W_\varepsilon.$$

Subtracting $v_\varepsilon + E_r$, whose absolute value is at most $2E_r$, and using the same Lipschitz comparison with Z , improves one of its two extremes. Hence, for a fixed $\theta \in (0, 1)$,

$$\operatorname{osc}_{B_{r/4}} Z \leq (1 - \theta) \operatorname{osc}_{B_r} Z + C(r + r^a). \quad (\text{B.25})$$

Iteration gives a uniform C^b bound for any fixed

$$0 < b < \min \left\{ a, \frac{-\log(1-\theta)}{\log 4} \right\}.$$

Applying the same conclusion to the reflected equation proves the estimate up to the constant Dirichlet face.

□

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